

THE BIG q -JACOBI FUNCTION TRANSFORM

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ABSTRACT. We give a detailed description of the resolution of the identity of a second order q -difference operator considered as an unbounded self-adjoint operator on two different Hilbert spaces. The q -difference operator and the two choices of Hilbert spaces naturally arise from harmonic analysis on the quantum group $SU_q(1, 1)$ and $SU_q(2)$. The spectral analysis associated to $SU_q(1, 1)$ leads to the big q -Jacobi function transform together with its Plancherel measure and inversion formula. The dual orthogonality relations give a one-parameter family of non-extremal orthogonality measures for the continuous dual q^{-1} -Hahn polynomials with $q^{-1} > 1$, and explicit sets of functions which complement these polynomials to orthogonal bases of the associated Hilbert spaces. The spectral analysis associated to $SU_q(2)$ leads to a functional analytic proof of the orthogonality relations and quadratic norm evaluations for the big q -Jacobi polynomials.

1. INTRODUCTION

Harmonic analysis on the compact quantum group $SU_q(2)$ has been studied by several authors, see for example [23], [19], [17], [14]. One possible approach is studying the restriction of the Haar functional and of the action of the quantum Casimir to “functions” in $SU_q(2)$ of a given, fixed bi- T -type, where T is the standard compact torus in $SU(2)$. This reduces the harmonic analysis to the spectral analysis of an explicit second order q -difference operator considered as unbounded linear operator on an explicit Hilbert space. The Hilbert space is the L^2 -space corresponding to the orthogonality measure of the little q -Jacobi polynomials, while the second order q -difference operator is diagonalized by the little q -Jacobi polynomials, so this leads to an interpretation of (the orthogonality relations of) the little q -Jacobi polynomials on $SU_q(2)$.

This approach was employed by Takeuchi, Masuda and Ueno [12] and Takeuchi [11] for the development of harmonic analysis on the non-compact quantum group $SU_q(1, 1)$, see also Vaksman and Korogodsky [15]. The restriction of the Casimir to functions on $SU_q(1, 1)$ of a given, fixed bi- T -type leads to the same second order q -difference operator as for $SU_q(2)$, while the restriction of a Haar functional on $SU_q(1, 1)$ to the given bi- T -type leads to a L^2 -space with respect to a discrete measure with unbounded support of the form $[0, \infty(z))_q = \{zq^k\}_{k \in \mathbb{Z}}$ ($z \neq 0$), where $0 < q < 1$. The spectral analysis of the second order q -difference operator considered as unbounded linear operator on this Hilbert space was developed in

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[12], [11]. It leads to the Plancherel formula and the inversion formula for the little q -Jacobi function transform.

It is well known that harmonic analysis on $SU_q(2)$ can be generalized by replacing the role of the torus T in $SU_q(2)$ by a “conjugate” T_t ($t \in \mathbb{R}$), which is defined in terms of twisted primitive elements in the quantized universal enveloping algebra $U_q(su(2))$, see [17]. In particular, considering the harmonic analysis on $SU_q(2)$ with respect to left T -types and right T_t -types leads to an interpretation of the big q -Jacobi polynomials on $SU_q(2)$, see [19]. Furthermore, the restriction of the Haar functional to functions of a given left T -type and right T_t -type can be computed directly and identified with the orthogonality measure of the big q -Jacobi polynomials, see [15] for the spherical case.

In this paper we develop the functional analytic aspects related to the harmonic analysis on $SU_q(1, 1)$ with respect to left T -types and right T_t -types. Concretely, we give a detailed description of the spectral properties of the second order q -difference operator L which is diagonalized by the big q -Jacobi polynomials, considered as an unbounded linear operator on a one-parameter family of Hilbert spaces. The Hilbert spaces are L^2 -spaces corresponding to explicit discrete measures with unbounded support $[-1, \infty(z))_q = \{-q^k\}_{k \in \mathbb{Z}_+} \cup \{zq^k\}_{k \in \mathbb{Z}}$ ($z > 0$). They can be interpreted for specific parameter values as the restriction of a Haar functional on $SU_q(1, 1)$ to functions of a given left T -type and right T_t -type.

We construct a one-parameter family of dense domains for which L is self-adjoint. With respect to one of these domains, we explicitly compute the resolution of the identity of L . This leads to the big q -Jacobi function transform, its Plancherel formula and its inversion formula. Furthermore, we show that the corresponding dual orthogonality relations lead to a one-parameter family of non-extremal orthogonality measures for the continuous dual q^{-1} -Hahn polynomials, and explicit sets of functions complementing these polynomials to orthogonal bases of the corresponding Hilbert spaces.

The little (respectively big) q -Jacobi function transform is a q -analogue of the Jacobi function transform on the interval $[0, \infty)$ (respectively $[-1, \infty)$), see e.g. Braaksma and Meulenbeld [4], Koornwinder [16] and references given there. In the classical setting, these two transforms are related by a dilation of the geometric parameter in which the end-point 0 of the interval $[0, \infty)$ is mapped onto the end-point -1 of the interval $[-1, \infty)$. The key point in our study of the big q -Jacobi function transform is to show that the limiting point 0 of the q -interval $[-1, \infty(z))_q$ does not play a special role in the spectral analysis, while the role of the end-point -1 is similar to the limiting point 0 for the little q -Jacobi function transform, see [12], [11].

We will report elsewhere in detail about the connection between the big q -Jacobi function transform and harmonic analysis on $SU_q(1, 1)$. Furthermore, we will report elsewhere on the Plancherel formula and the inversion formula for the Askey-Wilson function transform, which is associated to harmonic analysis on $SU_q(1, 1)$ with respect to left T_s -types and right T_t -types ($s, t \in \mathbb{R}$).

The organization of the paper is as follows.

In section 2 we introduce the second order q -difference operator L and the Hilbert spaces on which we consider L as an unbounded linear operator. Furthermore, we derive dense domains for which L is self-adjoint. The domains are given explicitly in terms of continuity and differentiability conditions of the functions at the origin.

In section 3 we derive criteria when it is possible to extend an arbitrary eigenfunction of L on the positive real axis to a global eigenfunction in such a way that the solution is “continuously differentiable” at the origin. We furthermore give, in terms of basic hypergeometric series, two explicit eigenfunctions which are continuously differentiable at the origin. One of them is the spherical function for the big q -Jacobi function transform.

In section 4 we introduce the asymptotically free eigenfunction of L on the positive real axis and we give the corresponding c -function expansion of the spherical function on the positive real axis.

In section 5 we extend for generic parameters the asymptotically free solution to a global eigenfunction of L in such a way, that the global eigenfunction is continuously differentiable at the origin. This leads to the c -function expansion of the spherical function on the whole support of the measure.

In section 6 we define the Green function of L in terms of the spherical function and the extended asymptotically free eigenfunction.

In section 7 we use the Green function to compute the continuous contribution to the resolution of the identity of L . Furthermore, we derive the Plancherel formula and the inversion formula for the continuous part of the big q -Jacobi function transform.

In section 8 we derive the discrete contribution of the resolution of the identity of L and we derive the orthogonality relations and the quadratic norm evaluations of the corresponding spherical functions, which are square integrable in these cases. We furthermore give a precise description of the spectrum of L .

In section 9 we state and prove the main results of the paper. We derive the Plancherel formula and the inversion formula for the big q -Jacobi function transform. Furthermore, we show that the dual orthogonality relations lead to a one-parameter family of non-extremal orthogonality measures for the continuous dual q^{-1} -Hahn polynomials, as well as explicit sets of functions which complement the polynomials to orthogonal bases of the corresponding Hilbert spaces.

Finally we derive in section 10 a functional analytic proof of the orthogonality relations and the quadratic norm evaluations for the big q -Jacobi polynomials. Proofs will only be sketched in this section, we mainly emphasize the differences between the compact setting, which corresponds to $SU_q(2)$ and the big q -Jacobi polynomials, and the non-compact setting, which corresponds to $SU_q(1, 1)$ and the big q -Jacobi functions.

Notations: We assume throughout the paper that $0 < q < 1$ is fixed. We follow the notation of Gasper and Rahman [7] concerning q -shifted factorials and basic hypergeometric series. We write

$$\theta(x_1, \dots, x_r) = \theta(x_1) \dots \theta(x_r)$$

for products of (renormalized) Jacobi theta-products $\theta(x) = (x, q/x; q)_\infty$. We write $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$ and $\mathbb{N} = \{1, 2, \dots\}$.

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2. THE SECOND ORDER q -DIFFERENCE OPERATOR

In this section we consider the second order q -difference operator L associated with the big q -Jacobi polynomials as an unbounded operator on a one-parameter family of Hilbert space. We determine suitable domains of definition for which L is self-adjoint.

The second order q -difference operator L depends on three parameters (a, b, c) . The corresponding one-parameter family of Hilbert spaces thus depends on four parameters (a, b, c, z) , where the parameter z labels the Hilbert spaces. In sections 2–9 we assume that the four parameters (a, b, c, z) satisfy $z > 0$ and $(a, b, c) \in V$, where

$$V = \{(a, b, c) \mid a, b, c > 0, \quad ab, ac, bc < 1\}, \quad (2.1)$$

unless explicitly stated otherwise. Observe that two of the three parameters a, b, c take their values in the open interval $(0, 1)$.

We define the q -interval I by

$$I = [-1, \infty(z))_q = \{-q^k \mid k \in \mathbb{Z}_+\} \cup \{zq^k \mid k \in \mathbb{Z}\},$$

which is regarded as a discrete q -analogue of the interval $[-1, \infty)$. We use the notation $(-1, \infty(z))_q$ for the q -interval $I \setminus \{-1\} = [-q, \infty(z))_q$. Let $\mathcal{F}(I)$ be the linear space of complex-valued functions $f : I \rightarrow \mathbb{C}$, and define a linear operator $L \in \text{End}_{\mathbb{C}}(\mathcal{F}(I))$ by

$$L = A(\cdot)(T_q - \text{Id}) + B(\cdot)(T_{q^{-1}} - \text{Id}), \quad (2.2)$$

with the q -shift operators defined by $(T_{q^{\pm 1}} f)(x) = f(q^{\pm 1}x)$, and with

$$A(x) = a^2 \left(1 + \frac{1}{abx}\right) \left(1 + \frac{1}{acx}\right), \quad B(x) = \left(1 + \frac{q}{bcx}\right) \left(1 + \frac{1}{x}\right). \quad (2.3)$$

Here $(Lf)(-1)$ is by definition given by

$$(Lf)(-1) = A(-1)(f(-q) - f(-1)), \quad f \in \mathcal{F}(I), \quad (2.4)$$

which is formally compatible with the fact that $B(-1) = 0$.

In the next lemma we rewrite the operator L as a second order operator in the q -difference operator D_q , which is defined by

$$(D_q f)(x) = \frac{f(x) - f(qx)}{(1 - q)x}.$$

Lemma 2.1. *Let $f \in \mathcal{F}(I)$, then*

$$(Lf)(x) = \begin{cases} p(x)(D_q(r(\cdot)D_q f))(q^{-1}x), & x \in (-1, \infty(z))_q, \\ \frac{-qp(x)r(x)}{(1 - q)x}(D_q f)(x), & x = -1, \end{cases} \quad (2.5)$$

where

$$\begin{aligned} p(x) &= \frac{(-abx, -acx; q)_\infty}{(-bcx, -qx; q)_\infty}, \\ r(x) &= \frac{(1-q)^2}{qbc} \frac{(-bcx, -qx; q)_\infty}{(-qabx, -qacx; q)_\infty}. \end{aligned} \quad (2.6)$$

Proof. Observe that p and r are well defined as functions on the q -interval I since $(a, b, c) \in V$. The formula for $x \in (-1, \infty(z))_q$ follows by observing that $p(\cdot)T_{q^{-1}}D_q r(\cdot)D_q$ is of the form (2.2) with A and B given by

$$A(x) = \frac{qp(x)r(x)}{(1-q)^2x^2}, \quad B(x) = \frac{q^2p(x)r(q^{-1}x)}{(1-q)^2x^2}.$$

By inserting the explicit functions p and r , we see that the corresponding A and B coincide with the ones given by (2.3). The formula for Lf in the point $x = -1$ follows now immediately from the definition (2.4) of $(Lf)(-1)$. \square

We have that $p(x) > 0$ for all $x \in I$ since $(a, b, c) \in V$. We can thus define the Hilbert space

$$\mathcal{H} = \{ f \in \mathcal{F}(I) \mid \|f\|^2 = \langle f, f \rangle < \infty \}, \quad (2.7)$$

where

$$\langle f, g \rangle = \int_{-1}^{\infty(z)} f(x) \overline{g(x)} \frac{d_q x}{p(x)}, \quad (2.8)$$

with the Jackson q -integral for $\alpha, \gamma \in \mathbb{C}^*$ defined by

$$\begin{aligned} \int_\alpha^\beta f(x) d_q x &= \int_0^\beta f(x) d_q x - \int_0^\alpha f(x) d_q x, \quad \beta = \gamma, \infty(\gamma) \\ \int_0^\gamma f(x) d_q x &= (1-q) \sum_{n=0}^{\infty} f(\gamma q^n) \gamma q^n, \\ \int_0^{\infty(\gamma)} f(x) d_q x &= (1-q) \sum_{n=-\infty}^{\infty} f(\gamma q^n) \gamma q^n, \end{aligned} \quad (2.9)$$

for functions f such that the sums converge absolutely. If $\alpha = q^k \gamma$ for certain $k \in \mathbb{Z}_+$, then the q -integral from α to γ can be defined for arbitrary functions f , since the q -integral reduces then to a finite sum.

We regard L as an unbounded operator on the Hilbert space \mathcal{H} . In the remainder of this section, we define a one-parameter family of dense subspaces $\mathcal{D} \subset \mathcal{H}$ such that L , with domain of definition \mathcal{D} , is self-adjoint. In order to determine the subspaces, it is convenient to consider the symmetry of L with respect to a truncated version of the inner product $\langle \cdot, \cdot \rangle$, which is defined for $k \in \mathbb{Z}_+$ and $l, m \in \mathbb{Z}$ with $l < m$ by

$$\begin{aligned} \langle f, g \rangle_{k;l,m} &= \left(\int_{-1}^{-q^{k+1}} + \int_{zq^{m+1}}^{zq^l} \right) f(x) \overline{g(x)} \frac{d_q x}{p(x)} \\ &= \sum_{n=0}^k f(-q^n) \overline{g(-q^n)} \frac{(1-q)q^n}{p(-q^n)} + \sum_{n=l}^m f(zq^n) \overline{g(zq^n)} \frac{(1-q)zq^n}{p(zq^n)}. \end{aligned}$$

Observe that for all $f, g \in \mathcal{H}$, we have

$$\lim_{\substack{k, m \rightarrow \infty \\ l \rightarrow -\infty}} \langle f, g \rangle_{k; l, m} = \langle f, g \rangle. \quad (2.10)$$

We have now the following lemma.

Lemma 2.2. *Let $f, g \in \mathcal{F}(I)$, $k \in \mathbb{Z}_+$ and $l, m \in \mathbb{Z}$ with $l < m$, then*

$$\langle Lf, g \rangle_{k; l, m} - \langle f, Lg \rangle_{k; l, m} = W(f, \bar{g})(zq^{l-1}) - W(f, \bar{g})(zq^m) + W(f, \bar{g})(-q^k),$$

with $\bar{f}(x) = \overline{f(x)}$ and with the Wronskian $W(f, g) \in \mathcal{F}(I)$ defined by

$$\begin{aligned} W(f, g)(x) &= \frac{qr(x)}{(1-q)x} (f(x)g(qx) - f(qx)g(x)) \\ &= qr(x)((D_q f)(x)g(x) - f(x)(D_q g)(x)) \end{aligned} \quad (2.11)$$

for all $x \in I$.

Proof. Since p and r (respectively A and B) are real valued on the q -interval I , we may restrict the proof to real valued functions f and g . Then we derive by a direct computation using Lemma 2.1 that

$$\begin{aligned} &\left\{ (Lf)(x)g(x) - f(x)(Lg)(x) \right\} \frac{(1-q)x}{p(x)} \\ &= \begin{cases} W(f, g)(q^{-1}x) - W(f, g)(x), & x \in (-1, \infty(z))_q, \\ -W(f, g)(-1), & x = -1. \end{cases} \end{aligned}$$

The lemma is now an easy consequence of this formula since the finite sums become telescoping. \square

In the remainder of this section, we use some standard terminology on unbounded linear operators, see [6, Chapter XII] and [20, Chapter 13]. Let $\mathcal{D} \subset \mathcal{H}$ be a dense linear space satisfying $L(\mathcal{D}) \subset \mathcal{H}$. Then \mathcal{D} may be considered as a domain of definition for the unbounded operator L on \mathcal{H} . Lemma 2.2 and (2.10) then show that (L, \mathcal{D}) is a densely defined symmetric operator if and only if for all $f, g \in \mathcal{D}$, the three limits

$$\lim_{l \rightarrow -\infty} W(f, \bar{g})(zq^{l-1}), \quad \lim_{m \rightarrow \infty} W(f, \bar{g})(zq^m), \quad \lim_{k \rightarrow \infty} W(f, \bar{g})(-q^k)$$

exist, and

$$\lim_{l \rightarrow -\infty} W(f, \bar{g})(zq^{l-1}) - \lim_{m \rightarrow \infty} W(f, \bar{g})(zq^m) + \lim_{k \rightarrow \infty} W(f, \bar{g})(-q^k) = 0.$$

The following lemma determines the behaviour of the Wronskian in the limit to infinity.

Lemma 2.3. *Let $f, g \in \mathcal{H}$, then $\lim_{m \rightarrow \infty} W(f, g)(zq^{-m}) = 0$.*

Proof. Using the formula

$$(xq^{1-m}; q)_\infty = (-x)^m q^{-m(m-1)/2} (1/x; q)_m (qx; q)_\infty, \quad x \in \mathbb{C}^*, \quad (2.12)$$

it follows that the behaviour of the weights of $\langle \cdot, \cdot \rangle$ at infinity is given by

$$\frac{(1-q)zq^{-m}}{p(zq^{-m})} = Ka^{-2m} (1 + \mathcal{O}(q^m)), \quad m \rightarrow \infty, \quad (2.13)$$

where K is the positive constant

$$K = (1 - q)z \frac{\theta(-bcz, -qz)}{\theta(-abz, -acz)}. \quad (2.14)$$

This implies that $\lim_{m \rightarrow \infty} a^{-m} f(zq^{-m}) = 0$ for all $f \in \mathcal{H}$. The proof follows now from the definition of the Wronskian (2.11) and the fact that

$$\frac{qr(zq^{-m})}{(1 - q)zq^{-m}} = Ka^{2-2m} (1 + \mathcal{O}(q^m)), \quad m \rightarrow \infty. \quad (2.15)$$

□

It follows from Lemma 2.3 that the domains $\mathcal{D} \subset \mathcal{H}$ for which (L, \mathcal{D}) is symmetric are determined by vanishing properties of the Wronskian at the origin. Before we define suitable domains of definition for L explicitly, we first introduce some convenient notations. For $f \in \mathcal{F}(I)$, we define

$$\begin{aligned} f(0^+) &= \lim_{k \rightarrow \infty} f(zq^k), & f(0^-) &= \lim_{k \rightarrow \infty} f(-q^k), \\ f'(0^+) &= \lim_{k \rightarrow \infty} (D_q f)(zq^k), & f'(0^-) &= \lim_{k \rightarrow \infty} (D_q f)(-q^k) \end{aligned} \quad (2.16)$$

provided that the limits exist. From now on we tacitly assume that the limits exist whenever we write $f(0^+)$, $f(0^-)$ etc. Let $\mathbb{T} = \{\alpha \in \mathbb{C} \mid |\alpha| = 1\}$ be the unit circle in the complex plane.

Definition 2.4. Let $\alpha \in \mathbb{T}$. We write $\mathcal{D}_\alpha \subset \mathcal{H}$ for the subspace of functions $f \in \mathcal{H}$ satisfying $Lf \in \mathcal{H}$, $f(0^+) = \alpha f(0^-)$ and $f'(0^+) = \alpha f'(0^-)$.

Observe that \mathcal{D}_α contains the functions with finite support, hence $\mathcal{D}_\alpha \subset \mathcal{H}$ is dense.

Lemma 2.5. Let $\alpha \in \mathbb{T}$.

- (i) There exists a function $f \in \mathcal{D}_\alpha$ such that $f(0^+) \neq 0$ and $(D_q f)(zq^k) = 0$, $(D_q f)(-q^l) = 0$ for $k, l \in \mathbb{Z}_+$.
- (ii) There exists a function $g \in \mathcal{D}_\alpha$ such that $g(0^+) = 0$ and $g'(0^+) \neq 0$.
- (iii) $\mathcal{D}_\alpha \neq \mathcal{D}_\beta$ for $\alpha, \beta \in \mathbb{T}$ with $\alpha \neq \beta$.

Proof. Part (iii) follows immediately from (i) and Definition 2.4. For the proof of (i), we observe first that the weights of $\langle \cdot, \cdot \rangle$ around zero behave like

$$\frac{(1 - q)q^k}{p(-q^k)} = \mathcal{O}(q^k), \quad \frac{(1 - q)zq^k}{p(zq^k)} = \mathcal{O}(q^k), \quad k \rightarrow \infty. \quad (2.17)$$

Define now the function $f \in \mathcal{F}(I)$ by $f(x) = 1$ if $-1 \leq x < 0$, $f(x) = \alpha$ if $0 < x \leq z$ and $f(x) = 0$ if $x > z$. Then $f \in \mathcal{H}$ follows from (2.17), and $Lf \in \mathcal{H}$ since $(Lf)(x) = 0$ if $x \notin \{z, q^{-1}z\}$. Furthermore, $f(0^+) = \alpha = \alpha f(0^-)$ and $f'(0^+) = 0 = \alpha f'(0^-)$, hence $f \in \mathcal{D}_\alpha$. By construction we have $f(0^+) \neq 0$ and $(D_q f)(zq^k) = 0$, $(D_q f)(-q^l) = 0$ for $k, l \in \mathbb{Z}_+$.

For the proof of (ii) we define the function $g \in \mathcal{F}(I)$ by $g(x) = x$ if $-1 \leq x < 0$, $g(x) = \alpha x$ if $0 < x \leq z$, and $g(x) = 0$ if $x > z$. Then $g \in \mathcal{H}$. Furthermore, $g(0^+) = 0 = \alpha g(0^-)$ and $g'(0^+) = \alpha = \alpha g'(0^-)$. So it remains to show that $Lg \in \mathcal{H}$. Using the explicit expression (2.2) for the q -difference operator L , we see that $(Lg)(-q^k) = \mathcal{O}(1)$, $(Lg)(zq^k) = \mathcal{O}(1)$ as $k \rightarrow \infty$. Combined with (2.17), it follows that $Lg \in \mathcal{H}$. □

Lemma 2.6. *Let $\alpha \in \mathbb{T}$, then (L, \mathcal{D}_α) is a symmetric operator.*

Proof. We have to show that

$$\langle Lf, g \rangle - \langle f, Lg \rangle = 0, \quad \forall f, g \in \mathcal{D}_\alpha.$$

By Lemma 2.2, Lemma 2.3, (2.10) and the fact that $h, Lh \in \mathcal{H}$ for $h \in \mathcal{D}_\alpha$, it suffices to show that $W(f, \bar{g})(0^+) = W(f, \bar{g})(0^-)$ for all $f, g \in \mathcal{D}_\alpha$. Since $r(0^+) = (1-q)^2/qbc = r(0^-)$, we have for all $f, g \in \mathcal{D}_\alpha$,

$$\begin{aligned} W(f, \bar{g})(0^+) &= \frac{(1-q)^2}{bc} (f'(0^+) \overline{g(0^+)} - f(0^+) \overline{g'(0^+)}) \\ &= \frac{(1-q)^2}{bc} \alpha \bar{\alpha} (f'(0^-) \overline{g(0^-)} - f(0^-) \overline{g'(0^-)}) = W(f, \bar{g})(0^-), \end{aligned}$$

as desired. \square

For $\alpha \in \mathbb{T}$, we write $(L^*, \mathcal{D}_\alpha^*)$ for the adjoint of the operator (L, \mathcal{D}_α) . Since (L, \mathcal{D}_α) is a symmetric operator, we have $(L, \mathcal{D}_\alpha) \subset (L^*, \mathcal{D}_\alpha^*)$.

Recall that L was initially defined as a linear operator on the linear space $\mathcal{F}(I)$ of complex-valued functions on I . In particular, L is well defined as a linear map $L : \mathcal{D}_\alpha^* \rightarrow \mathcal{F}(I)$ by restriction. We claim that

$$L^* = L|_{\mathcal{D}_\alpha^*}. \quad (2.18)$$

To prove the claim, we observe that

$$\langle Lf, g \rangle = \langle f, Lg \rangle$$

for functions $f, g \in \mathcal{F}(I)$ such that f has finite support (compare with the proof of Lemma 2.2). For $g \in \mathcal{D}_\alpha^*$ this implies that $\langle f, Lg \rangle = \langle f, L^*g \rangle$ for functions f with finite support. Applying this formula with a non-zero function f having support in one point $x \in I$, we arrive at $(Lg)(x) = (L^*g)(x)$. The claim (2.18) follows, since $x \in I$ can be chosen arbitrarily.

Proposition 2.7. *Let $\alpha \in \mathbb{T}$, then (L, \mathcal{D}_α) is self-adjoint, i.e. $\mathcal{D}_\alpha = \mathcal{D}_\alpha^*$.*

Proof. Since (L, \mathcal{D}_α) is symmetric, it suffices to prove the inclusion $\mathcal{D}_\alpha^* \subset \mathcal{D}_\alpha$. Let $g \in \mathcal{D}_\alpha^*$. Then by (2.18), $Lg = L^*g \in \mathcal{H}$. For any $f \in \mathcal{D}_\alpha$ we have, using (2.18), Lemma 2.2, Lemma 2.3 and (2.10), that

$$0 = \langle Lf, g \rangle - \langle f, L^*g \rangle = W(f, \bar{g})(0^-) - W(f, \bar{g})(0^+).$$

Now using Lemma 2.5, one derives from the existence of the limits

$$\begin{aligned} W(f, \bar{g})(0^+) &= \frac{(1-q)^2}{bc} \lim_{k \rightarrow \infty} ((D_q f)(zq^k) \overline{g(zq^k)} - f(zq^k) \overline{(D_q g)(zq^k)}) \\ W(f, \bar{g})(0^-) &= \frac{(1-q)^2}{bc} \lim_{k \rightarrow \infty} ((D_q f)(-q^k) \overline{g(-q^k)} - f(-q^k) \overline{(D_q g)(-q^k)}) \end{aligned}$$

for all $f \in \mathcal{D}_\alpha$ that $g(0^+) = \alpha g(0^-)$ and $g'(0^+) = \alpha g'(0^-)$. It follows that $g \in \mathcal{D}_\alpha$, as desired. \square

Observe that the operator $L \in \text{End}(\mathcal{F}(I))$ preserves the subspace of functions $f \in \mathcal{F}(I)$ with support in $I_- = [-1, 0]_q = \{-q^k\}_{k \in \mathbb{Z}_+}$, as well as the subspace of functions $f \in \mathcal{F}(I)$ with support in $I_+ = [0, \infty(z))_q = \{zq^l\}_{l \in \mathbb{Z}}$. We denote L_- (respectively L_+) for the restriction of L to functions $f \in \mathcal{F}(I)$ with support in I_- (respectively I_+) and we write $\mathcal{H} = \mathcal{H}^- \oplus \mathcal{H}^+$ for the orthogonal direct sum decomposition where \mathcal{H}^\pm are the functions $f \in \mathcal{H}$ with support in I_\pm .

We end this section by indicating the relation of this splitting of L with the possible choices of domains for L . The following results will not be used in the remainder of the paper, so we omit detailed proofs.

We start with the symmetric operator $(L_+, \mathcal{D}_{fin}^+)$ on \mathcal{H}_+ , where $\mathcal{D}_{fin}^+ \subset \mathcal{H}_+$ is the subspace of functions with finite support. Its minimal closure is given by (L, \mathcal{D}_0^+) , where

$$\mathcal{D}_0^+ = \{f \in \mathcal{H}^+ \mid L_+f \in \mathcal{H}^+, f(0^+) = f'(0^+) = 0\},$$

and the deficiency indices of (L, \mathcal{D}_0^+) are $(1, 1)$.

The spectral properties of the self-adjoint extensions of (L_+, \mathcal{D}_0^+) are highly sensitive with respect to the choice of self-adjoint extension. Furthermore, the spectral analysis of L_+ with respect to a fixed choice of self-adjoint extension seems to result in a rather implicit description of the spectrum and of the resolution of the identity.

In order to obtain better spectral properties, we “blow up” the Hilbert space \mathcal{H}_+ to $\mathcal{H} = \mathcal{H}_- \oplus \mathcal{H}_+$ and we regard L now as an unbounded operator on \mathcal{H} with domain \mathcal{D}_{fin} given by the functions in \mathcal{H} with finite support. Its minimal closure is given by (L, \mathcal{D}_0) , where

$$\mathcal{D}_0 = \{f \in \mathcal{H} \mid Lf \in \mathcal{H}, f(0^+) = f(0^-) = 0 = f'(0^-) = f'(0^+)\}.$$

The closed, symmetric operator (L, \mathcal{D}_0) has deficiency indices $(2, 2)$. By the increase of deficiency indices we have a larger choice of self-adjoint extensions for (L, \mathcal{D}_0) than for (L_+, \mathcal{D}_0^+) . In particular, the self-adjoint extensions (L, \mathcal{D}_α) ($\alpha \in \mathbb{T}$) are self-adjoint extensions for which $(L_+, \mathcal{D}_\alpha \cap \mathcal{H}_+)$ are *not* self-adjoint.

It turns out that the spectral analysis of (L, \mathcal{D}_α) is essentially independent of $\alpha \in \mathbb{T}$, so we restrict attention in this paper to the unbounded self-adjoint operator L on \mathcal{H} with domain of definition $\mathcal{D} = \mathcal{D}_1 \subset \mathcal{H}$. In particular, any function $f \in \mathcal{D}$ is *continuously differentiable* at the origin, i.e. f satisfies $f(0^+) = f(0^-)$ and $f'(0^+) = f'(0^-)$.

In the next sections we show that (L, \mathcal{D}) is a self-adjoint extension of (L, \mathcal{D}_0) for which the spectral analysis can be derived in a very explicit manner. The reason is that for a given generic eigenvalue, one has two linear independent eigenfunctions of L which are explicitly given in terms of basic hypergeometric series and which are continuously differentiable at the origin. The corresponding Wronskian, as well as their Wronskian with the asymptotically free solutions of the corresponding eigenvalue equation, can be computed explicitly. This allows us to derive the spectral properties in a very explicit manner.

Let us finally make some remarks on the “blowing up” procedure of the Hilbert space \mathcal{H}_+ as described in the previous paragraphs. It is a well known principle in harmonic analysis that the spectral properties of the unbounded self-adjoint operator under consideration is essentially determined by the behaviour of the eigenfunctions at infinity (along the support of the measure). The fact that \mathcal{H}_- is a L^2 -space of functions supported on a compact space implies that the situation at infinity does not alter in the enlarged setting. Hence, the spectral properties of the extension of L_+ to the unbounded operator L on \mathcal{H} should be essentially determined by spectral properties of L_+ . We will justify this principle in this paper by deriving most of the spectral properties of L from the spectral properties of L_+ .

The blowing up of the Hilbert space \mathcal{H}_+ is canonical in the sense that the extra piece \mathcal{H}_- added to the Hilbert space \mathcal{H}_+ is essentially determined by the following three properties:

- (i) The support of the measure of \mathcal{H}^- is a *bounded* q -interval,
- (ii) The operator (L, \mathcal{D}_{fin}^-) on \mathcal{H}^- is symmetric, where \mathcal{D}_{fin}^- consists of functions in \mathcal{H}^- with finite support,
- (iii) The weight function is continuous at the origin.

Indeed, the condition that the support of the measure of \mathcal{H}^- is a bounded q -interval implies that the support is of the form $[y, 0]_q = \{yq^k\}_{k \in \mathbb{Z}_+}$ with $B(y) = 0$. This leads to the possibilities $y = -1$ or $y = -bc/q$. Choosing $y = -1$, we derive from condition (ii) that the weight function corresponding to \mathcal{H}^- is given by $1/p(\cdot)$ up to a positive constant, while condition (iii) implies that the constant is one. Since $p(x; a, b, c) = p(bc x/q; a, q/b, q/c)$ and a similar property holds for L , see Remark 3.4, we may take $y = -1$ without loss of generality.

3. EIGENFUNCTIONS OF L

In sections 3–9 we consider L with domain of definition $\mathcal{D} = \mathcal{D}_1$. In particular, any function f in the domain of definition \mathcal{D} is continuously differentiable at the origin. We set $f(0)$ (respectively $f'(0)$) for the common limits $f(0^\pm)$ (respectively $f'(0^\pm)$).

Furthermore, we need to restrict sometimes the choice of parameters $(a, b, c) \in V$ to a dense subdomain V_z^{gen} , which is defined by

$$V_z^{gen} = \{(a, b, c) \in V \mid a^2, b^2, c^2, ab, ac, bc, a/b, a/c, a^2 b^2 c^2 z^2 \notin \{q^k\}_{k \in \mathbb{Z}}\}. \quad (3.1)$$

In this section we consider eigenfunctions of the linear operators L and L_\pm , and study their behaviour at the origin. We write $(-1, 0]_q = I_- \setminus \{-1\} = [-q, 0]_q$, and define for $\mu \in \mathbb{C}$ the linear spaces

$$\begin{aligned} V_\mu^\pm &= \{f : I_\pm \rightarrow \mathbb{C} \mid L_\pm f = \mu f \text{ on } I_\pm\}, \\ \hat{V}_\mu^- &= \{f : I_- \rightarrow \mathbb{C} \mid L_- f = \mu f \text{ on } (-1, 0]_q\}, \\ V_\mu &= \{f \in \mathcal{F}(I) \mid Lf = \mu f \text{ on } (-1, \infty(z))_q, f \text{ cont. differentiable in } 0\}. \end{aligned} \quad (3.2)$$

Observe that $V_\mu^- \subset \hat{V}_\mu^-$.

Lemma 3.1. *Let $\mu \in \mathbb{C}$.*

- (i) $\dim(V_\mu^+) = 2$, $\dim(V_\mu^-) = 1$ and $\dim(\hat{V}_\mu^-) = 2$.
- (ii) If $f_1, f_2 \in V_\mu^+$ (respectively $f_1, f_2 \in \hat{V}_\mu^-$), then $W(f_1, f_2)$ is constant on I_+ (respectively constant on I_-).
- (iii) If $f_1, f_2 \in V_\mu$, then $W(f_1, f_2) \in \mathcal{F}(I)$ is constant on I .

Proof. (i) We start with computing $\dim(V_\mu^-)$. The coefficient $A(x)$ in the expression (2.2) of L is non-zero for $x \in I_-$ since $(a, b, c) \in V$. It follows that solutions of $L_- f = \mu f$ on I_- are in one to one correspondence with solutions $(a_k)_{k \in \mathbb{Z}_+}$ of a recurrence relation of the form

$$\mu a_k = \alpha_k a_{k-1} + \beta_k a_k + \gamma_k a_{k+1}, \quad \gamma_k \neq 0, \quad k \in \mathbb{Z}_+$$

with $a_{-1} = 0$ by definition. The correspondence is obtained by associating the sequence $a_k = f(-q^k)$ ($k \in \mathbb{Z}_+$) to $f \in V_\mu^-$. The coefficients of the recurrence relation are then given by $\alpha_k = -B(-q^k)$, $\beta_k = -A(-q^k) - B(-q^k)$ and $\gamma_k =$

$A(-q^k)$. Any solution of such a recurrence relation is uniquely determined by $a_0 = f(-1) \in \mathbb{C}$, hence $\dim(V_\mu^-) = 1$.

Then $\dim(\hat{V}_\mu^-) = 2$ follows from the fact that functions $f \in \hat{V}_\mu^-$ are in one to one correspondence with solutions $(a_k)_{k \in \mathbb{Z}_+}$ of the recurrence relation

$$\mu a_k = \alpha_k a_{k-1} + \beta_k a_k + \gamma_k a_{k+1}, \quad \gamma_k \neq 0, \quad k \in \mathbb{N}.$$

Finally, in order to show that $\dim(V_\mu^-) = 2$, observe that solutions of $L_+ f = \mu f$ on I_+ are in one to one correspondence with solutions $(a_k)_{k \in \mathbb{Z}}$ of a double infinite recurrence relation of the form

$$\mu a_k = \alpha_k a_{k-1} + \beta_k a_k + \gamma_k a_{k+1}, \quad \alpha_k \neq 0, \gamma_k \neq 0, \quad k \in \mathbb{Z},$$

since $A(x)$ and $B(x)$ (2.3) in the expression (2.2) of L are non-zero for $x \in I_+$. This is a two-dimensional space.

(ii) Let $f_1, f_2 \in \mathcal{F}(I)$. Using Lemma 2.1, the product rule for the q -derivative

$$(D_q(fg))(x) = (D_q f)(x)g(x) + f(qx)(D_q g)(x), \quad (3.3)$$

and the second equality of (2.11), we have for all $x \in (-1, \infty(z))_q$,

$$\begin{aligned} q^{-1}p(x)(D_q W(f_1, f_2))(q^{-1}x) \\ &= (Lf_1)(x)f_2(x) + p(x)r(q^{-1}x)(D_q f_1)(q^{-1}x)(D_q f_2)(q^{-1}x) \\ &\quad - (Lf_2)(x)f_1(x) - p(x)r(q^{-1}x)(D_q f_2)(q^{-1}x)(D_q f_1)(q^{-1}x) \\ &= (Lf_1)(x)f_2(x) - (Lf_2)(x)f_1(x). \end{aligned} \quad (3.4)$$

If $f_1, f_2 \in \hat{V}_\mu^-$, then it follows from (3.4) that $(D_q W(f_1, f_2))(x) = 0$ for all $x \in (-1, 0]_q$ since $p(x) \neq 0$ ($x \in I_-$), hence $W(f_1, f_2)$ is constant on I_- . A similar argument shows that $W(f_1, f_2)$ is constant on I_+ if $f_1, f_2 \in V_\mu^+$.

(iii) Let $f_1, f_2 \in V_\mu$. The continuously differentiability of the f_j 's at the origin yields that $W(f_1, f_2)(0^+) = W(f_1, f_2)(0^-)$. Indeed, both sides are equal to $qr(0)(f_1'(0)f_2(0) - f_1(0)f_2'(0))$ (compare with the proof of Lemma 2.6). The result follows now from (ii). \square

Observe that $f|_{I_-} \in \hat{V}_\mu^-$ and $f|_{I_+} \in V_\mu^+$ for $f \in V_\mu$.

Proposition 3.2. *We have $\dim(V_\mu) \leq 2$. Furthermore, the following three statements are equivalent:*

- (i) *The linear map $f \mapsto f|_{I_-} : V_\mu \rightarrow \hat{V}_\mu^-$ is a bijection.*
- (ii) *The linear map $f \mapsto f|_{I_+} : V_\mu \rightarrow V_\mu^+$ is a bijection.*
- (iii) *$\dim(V_\mu) = 2$.*

Proof. Suppose that $\dim(V_\mu) \geq 2$, then we claim that the map $f \mapsto f|_{I_+} : V_\mu \rightarrow V_\mu^+$ is injective. Suppose that the map is not injective. Then there exist two linearly independent functions $f_1, f_2 \in V_\mu$ such that $f_1|_{I_+}, f_2|_{I_+} \in V_\mu^+$ are linearly dependent. The Wronskian $W(f_1, f_2)$ is not identically zero as function on I by the linear independence of f_1 and f_2 . Since $W(f_1, f_2)$ is constant on I by Lemma 3.1(iii), it follows that $W(f_1, f_2)$ is not identically zero on I_+ , which contradicts the linear dependence of $f_1|_{I_+}$ and $f_2|_{I_+}$.

So if $\dim(V_\mu) \geq 2$, then it follows from Lemma 3.1(i) and from the injectivity of the map $f \mapsto f|_{I_+} : V_\mu \rightarrow V_\mu^+$ that $\dim(V_\mu) = 2$. This proves that $\dim(V_\mu) \leq 2$ for all $\mu \in \mathbb{C}$, and it proves the implication (iii) \Rightarrow (ii). The implication (iii) \Rightarrow (i)

is proved in a similar manner, while the implications (i) \Rightarrow (iii) and (ii) \Rightarrow (iii) are immediate consequences of Lemma 3.1(i). \square

Corollary 3.3. *If $\dim(V_\mu) = 2$, then any $g \in \hat{V}_\mu^-$ (respectively $g \in V_\mu^+$) extends uniquely to a function $g \in V_\mu$.*

In the remainder of this section we introduce two explicit functions which live in V_μ . We define for $\gamma \in \mathbb{C}^*$,

$$\mu(\gamma) = -1 - a^2 + a(\gamma + \gamma^{-1}), \quad (3.5)$$

and we write $X_n = (bcx^2/q)^n (-q/bcx, -1/x; q)_n g_\gamma(xq^{-n})$ where g_γ is a function satisfying $(Lg_\gamma)(xq^{-n}) = \mu(\gamma)g_\gamma(xq^{-n})$. Then the X_n satisfy the three term recurrence relation [9, (2.1)] with parameters (A, B, C, D, z) in [9, (2.1)] given by $A = -q/bcx$, $B = -1/x$, $C = -q/acx$, $D = -q/abx$ and

$$z = \frac{abcx^2}{q}(\gamma + \gamma^{-1}) = \frac{q}{ABCD\lambda_\pm} + \lambda_\pm, \quad \lambda_\pm = \frac{abcx^2}{q}\gamma^{\pm 1}.$$

In [9] it was shown that the three term recurrence relation [9, (2.1)] is the one satisfied by the associated continuous dual q -Hahn polynomials. Furthermore, several explicit solutions of the recurrence relation [9, (2.1)] were derived explicitly in [9, section 2]. In particular, the explicit solution [9, (2.13)] of the recurrence relation [9, (2.1)] implies that

$$\phi_\gamma(x) = \phi_\gamma(x; a, b, c) = {}_3\phi_2 \left(\begin{matrix} a\gamma, a/\gamma, -1/x \\ ab, ac \end{matrix}; q, -bcx \right), \quad |bcx| < 1 \quad (3.6)$$

satisfies $(L\phi_\gamma)(x) = \mu(\gamma)\phi_\gamma(x)$ for $x \in (-1, \infty(z))_q$ with $|x| < q/bc$. Observe that ϕ_γ is well defined since $(a, b, c) \in V$.

The explicit solution [9, (2.13)] of the recurrence relation [9, (2.1)] implies that

$$\psi_\gamma(x; a, b, c) = {}_3\phi_2 \left(\begin{matrix} a\gamma, a/\gamma, -q/bcx \\ qa/b, qa/c \end{matrix}; q, -qx \right), \quad |qx| < 1 \quad (3.7)$$

satisfies $(L\psi_\gamma)(x) = \mu(\gamma)\psi_\gamma(x)$ for $x \in (-1, \infty(z))_q$ with $|x| < 1$ when $(a, b, c) \in V_z^{gen}$.

Since $bc < 1$, we have that ϕ_γ and ψ_γ are well defined on I_- and that they are solutions of $Lg = \mu(\gamma)g$ on $(-1, 0]_q$. Since ϕ_γ and ψ_γ are solutions of $(Lg)(x) = \mu(\gamma)g(x)$ for $x \in I_+$ with $x < 1$, it follows that ϕ_γ and ψ_γ uniquely extend to functions on I satisfying the eigenvalue equation $Lg = \mu(\gamma)g$ on $(-1, \infty(z))_q$, cf. the proof of Lemma 3.1(i). For given $x \in I$, $\phi_\gamma(x)$ and $\psi_\gamma(x)$ depend analytically on $\gamma \in \mathbb{C}^*$ and are invariant under $\gamma \leftrightarrow \gamma^{-1}$.

The extension of ϕ_γ to a function on I can also be obtained by the transformation formula [7, (3.2.10)], which yields

$$\phi_\gamma(x) = \frac{(a\gamma, bc, -abcx/\gamma; q)_\infty}{(ab, ac, -bcx; q)_\infty} {}_3\phi_2 \left(\begin{matrix} b/\gamma, c/\gamma, -bcx \\ bc, -abcx/\gamma \end{matrix}; q, a\gamma \right). \quad (3.8)$$

This gives a single valued analytic continuation for $\phi_\gamma(x)$ for $\gamma \in \mathbb{C}^*$ with $|\gamma| < a^{-1}$ to $x \in \mathbb{C} \setminus (-\infty, -1/bc]$. Observe that $(-\infty, -1/bc] \cap I = \emptyset$ since $(a, b, c) \in V$. If $a < 1$, then the analytic continuation for $|\gamma| \geq a^{-1}$ can be obtained by replacing

γ by γ^{-1} in the right hand side of (3.8). If $a \geq 1$, then $b < 1$, and we can use [7, (3.2.7)] to obtain

$$\phi_\gamma(x) = \frac{(b\gamma, -abcx/\gamma; q)_\infty}{(ab, -bcx; q)_\infty} {}_3\phi_2 \left(\begin{matrix} a/\gamma, c/\gamma, -acx \\ ac, -abcx/\gamma \end{matrix}; q, b\gamma \right), \quad (3.9)$$

which yields then the explicit expression for the analytic continuation of ϕ_γ ($\gamma \in \mathbb{C}^*$) in a similar manner as for $a < 1$. A similar remark holds for the solution ψ_γ .

Remark 3.4. If $f(\cdot) = f(\cdot; a, b, c)$ satisfies the eigenvalue equation $Lf = \mu(\gamma)f$, then $g(x) = f(bcx/q; a, q/b, q/c)$ satisfies the same eigenvalue equation. It is easy to check using the explicit expressions (3.6) and (3.7) for ϕ_γ and ψ_γ that the solutions ϕ_γ and ψ_γ are interchanged by this symmetry, i.e.

$$\psi_\gamma(x; a, b, c) = \phi_\gamma(bcx/q; a, q/b, q/c). \quad (3.10)$$

From (3.6) we see that $\phi_\gamma(-1) = 1$ and by direct computation we have

$$(D_q \phi_\gamma)(x; a, b, c) = \frac{bc\mu(\gamma)}{(1-q)(1-ab)(1-ac)} \phi_\gamma(x; qa, b, c), \quad x \in I. \quad (3.11)$$

It follows that

$$(D_q \phi_\gamma)(-1) = \frac{bc\mu(\gamma)}{(1-q)(1-ab)(1-ac)}, \quad (3.12)$$

so $(L\phi_\gamma)(-1) = \mu(\gamma)\phi_\gamma(-1)$ by (2.4). We conclude that $\phi_\gamma|_{I_-} \in V_{\mu(\gamma)}^-$ for all $\gamma \in \mathbb{C}^*$. For ψ_γ we have the formula

$$(D_q \psi_\gamma)(x; a, b, c) = \frac{q\mu(\gamma)}{(1-q)(1-qa/b)(1-qa/c)} \psi_\gamma(x; qa, b, c). \quad (3.13)$$

We stress already the fact that in general we have $(L\psi_\gamma)(-1) \neq \mu(\gamma)\psi_\gamma(-1)$, so that $\psi_\gamma|_{I_-} \in \hat{V}_{\mu(\gamma)} \setminus V_{\mu(\gamma)}$, see Corollary 5.2(iii).

Lemma 3.5. *Let $\gamma \in \mathbb{C}^*$. The function $\phi_\gamma \in \mathcal{F}(I)$ is continuously differentiable at the origin, i.e. $\phi_\gamma \in V_{\mu(\gamma)}$. In fact, we have*

$$\begin{aligned} \phi_\gamma(0) &= {}_2\phi_2 \left(\begin{matrix} a\gamma, a/\gamma \\ ab, ac \end{matrix}; q, bc \right), \\ \phi'_\gamma(0) &= \frac{bc\mu(\gamma)}{(1-q)(1-ab)(1-ac)} {}_2\phi_2 \left(\begin{matrix} qa\gamma, qa/\gamma \\ qab, qac \end{matrix}; q, bc \right). \end{aligned} \quad (3.14)$$

For $(a, b, c) \in V_z^{gen}$ we have $\psi_\gamma \in V_{\mu(\gamma)}$ and

$$\psi_\gamma(0; a, b, c) = \phi_\gamma(0; a, q/b, q/c), \quad \psi'_\gamma(0; a, b, c) = \frac{bc}{q} \phi'_\gamma(0; a, q/b, q/c),$$

where we have extended the definition of $\phi_\gamma(0)$ and $\phi'_\gamma(0)$ to generic parameters $(a, b, c) \in \mathbb{C}^{\times 3}$ by analytic continuation of the right hand sides of (3.14).

Proof. The proof for ϕ_γ is a direct consequence of (3.11) and the explicit expression (3.6) for ϕ_γ . The proof for ψ_γ follows then from (3.7) and (3.13). \square

In section 5 we will evaluate the Wronskian $W(\psi_\gamma, \phi_\gamma)$ explicitly, see Proposition 5.1. In particular, this will give explicit criteria on the spectral parameter γ for which we have $\dim(V_{\mu(\gamma)}) = 2$, i.e. for which Corollary 3.3 is applicable. But first we need to study yet another solution of the eigenvalue equation $Lg = \mu(\gamma)g$, the so called asymptotic solution.

4. THE ASYMPTOTIC SOLUTION

In this section we determine the asymptotic solution Φ_γ of the eigenvalue equation $L_+g = \mu(\gamma)g$ on I_+ . We furthermore determine the c -function expansion of ϕ_γ on I_+ , i.e. we write $\phi_\gamma|_{I_+}$ explicitly as linear combination of the asymptotic solutions Φ_γ and $\Phi_{\gamma^{-1}}$.

We define singular sets by

$$S_{sing}^+ = \{\pm q^{-\frac{1}{2}k}\}_{k \in \mathbb{N}}, \quad S_{sing} = \{\pm q^{\frac{1}{2}k}\}_{k \in \mathbb{Z}}, \quad (4.1)$$

and we write S_{reg}^+ , S_{reg} for the complements of these singular sets in \mathbb{C}^* . The c -function expansion of ϕ_γ on I_+ which we derive in this section, holds for $\gamma \in S_{reg}$.

We consider for $y > 0$ the minimal solution [9, (2.26)] of the three term recurrence relation [9, (2.1)] with corresponding parameters (A, B, C, D, λ_-) given by

$$A = -q/bcy, \quad B = -q/aby, \quad C = -q/acx, \quad D = -1/y, \quad \lambda_- = \frac{abcy^2}{q}.$$

This leads for $\gamma \in S_{reg}^+$ with $|\gamma| < a^{-1}$ to the following explicit solution $\Phi_\gamma^y(x) = \Phi_\gamma^y(x; a, b, c)$ of $Lf = \mu(\gamma)f$ on $[0, \infty(y))_q$,

$$\begin{aligned} \Phi_\gamma^y(x) &= \frac{(-q\gamma/ax, -q^2\gamma/abcx, a\gamma; q)_\infty}{(-q/abx, -q/acx, q\gamma^2; q)_\infty} \\ &\cdot (a\gamma)^{-k} {}_3\phi_2 \left(\begin{matrix} q\gamma/a, -q/abx, -q/acx \\ -q\gamma/ax, -q^2\gamma/abcx \end{matrix} ; q, a\gamma \right), \quad x = yq^k. \end{aligned} \quad (4.2)$$

For given $x \in [0, \infty(y))_q$, we have a single valued analytic extension of $\Phi_\gamma^y(x)$ to $\gamma \in S_{reg}^+$. Indeed, for $x \in [0, \infty(y))_q$ with $x > bc/q$ we can apply [7, (3.2.7)] to arrive at

$$\begin{aligned} \Phi_\gamma^y(x) &= \frac{(-q/bcx, -q\gamma/ax; q)_\infty}{(-q/abx, -q/acx; q)_\infty} \\ &\cdot (a\gamma)^{-k} {}_3\phi_2 \left(\begin{matrix} q\gamma/a, b\gamma, c\gamma \\ -q\gamma/ax, q\gamma^2 \end{matrix} ; q, -q/bcx \right), \quad x = yq^k, \end{aligned} \quad (4.3)$$

which gives the analytic extension in this case. For $x \in [0, \infty(y))_q$ with $y \leq bc/q$ we can use the eigenvalue equation $L\Phi_\gamma^y = \mu(\gamma)\Phi_\gamma^y$ on $[0, \infty(y))_q$ to derive the analytic extension for $\Phi_\gamma^y(x)$ from the analytic extension of $\Phi_\gamma^y(u)$ with $u > bc/q$. It follows from (4.3) that the asymptotics to infinity of Φ_γ^y is given by

$$\Phi_\gamma^y(yq^{-m}) = (a\gamma)^m (1 + \mathcal{O}(q^m)), \quad m \rightarrow \infty. \quad (4.4)$$

Definition 4.1. Let $\gamma \in S_{reg}^+$. We call $\Phi_\gamma(\cdot; a, b, c) = \Phi_\gamma^z(\cdot; a, b, c)$ the asymptotic solution of $L_+f = \mu(\gamma)f$ on $I_+ = [0, \infty(z))_q$.

Definition 4.1 is justified by the following lemma.

Lemma 4.2. Let $\mu \in \mathbb{C}^*$ and $K = \{zq^{-k}\}_{k \in \mathbb{Z}_+} \subset I_+$. Set

$$M_\mu = \{h : I_+ \rightarrow \mathbb{C} \mid L_+h = \mu h \text{ on } I_+ \text{ and } \|h_K\|^2 < \infty\},$$

where $h_K \in \mathcal{F}(I)$ is defined to be equal to zero on $I \setminus K$ and to be equal to h on K . Then $M_\mu = \text{span}\{\Phi_\gamma\}$ for all $\mu \in \mathbb{C} \setminus \mathbb{R}$, where $\gamma \in \mathbb{C}^*$ is the unique non-zero complex number such that $\mu = \mu(\gamma)$ and $|\gamma| < 1$.

Proof. First of all, observe that if $\gamma \in \mathbb{C}^*$ and $|\gamma| < 1$, then $\gamma \in S_{reg}^+$, hence Φ_γ is well defined.

Let $\mu \in \mathbb{C} \setminus \mathbb{R}$ and let $\gamma \in \mathbb{C}^*$, $|\gamma| < 1$ be such that $\mu = \mu(\gamma)$. Using the asymptotics (2.13) for the weights of the inner product $\langle \cdot, \cdot \rangle$, it follows from (4.4) that $\Phi_\gamma \in M_{\mu(\gamma)}$ because $|\gamma| < 1$.

It remains to show that $\dim(M_\mu) = 1$. For the proof we use some well known results from the theory of the classical moment problem, see for instance [2] or [21]. If $f : I_+ \rightarrow \mathbb{C}$ satisfies $L_+f = \mu f$ on I_+ , then by setting

$$a_k = f(zq^{-k}) \sqrt{\frac{(1-q)zq^{-k}}{p(zq^{-k})}}, \quad k \in \mathbb{Z},$$

we see that $(a_k)_{k \in \mathbb{Z}}$ satisfies the recurrence relation

$$\alpha_{k-1}a_{k-1} + \beta_k a_k + \alpha_k a_{k+1} = \mu a_k \quad (4.5)$$

with

$$\alpha_k = a \sqrt{\left(1 + \frac{q^{k+1}}{abz}\right) \left(1 + \frac{q^{k+1}}{acz}\right) \left(1 + \frac{q^{k+1}}{bcz}\right) \left(1 + \frac{q^k}{z}\right)}$$

and $\beta_k = -A(zq^{-k}) - B(zq^{-k})$. It follows from [21, Corollary 4.5] and from the fact that the sequence $(\alpha_k)_{k \in \mathbb{Z}_+}$ is bounded that the Hamburger moment problem corresponding to the recurrence relation (4.5) for $k \in \mathbb{Z}_+$ is determined. By [21, Theorem 3] this implies that $\dim(M_\mu) = 1$, as desired. \square

It follows from Lemma 4.2 that

$$\Phi_\gamma^{bcz/q}(bcx/q; a, q/b, q/c) = \Phi_\gamma(x; a, b, c), \quad x \in I_+ \quad (4.6)$$

for parameters $(a, b, c) \in V$ such that $(a, q/b, q/c) \in V$. Indeed, both sides of (4.6), considered as function of $x \in I_+$, are solutions of $L_+f = \mu(\gamma)f$ on I_+ by Remark 3.4, and they have the same asymptotics to infinity by (4.4). Formula (4.6) is also obvious from the explicit expression (4.2) for Φ_γ^y .

Lemma 4.3. *For $\gamma \in S_{reg}$ and $x \in I_+$, we have*

$$W(\Phi_\gamma, \Phi_{\gamma^{-1}})(x) = aK(\gamma - \gamma^{-1}) \neq 0,$$

where K is the positive constant defined by (2.14). In particular, $\{\Phi_\gamma, \Phi_{\gamma^{-1}}\}$ is a basis of $V_{\mu(\gamma)}^+$ when $\gamma \in S_{reg}$.

Proof. The explicit expression for the Wronskian follows by computing the limit $\lim_{m \rightarrow \infty} W(\Phi_\gamma, \Phi_{\gamma^{-1}})(zq^{-m})$ using the first expression of (2.11) and the formulas (2.15) and (4.4). Since the Wronskian is non-zero, it follows that Φ_γ and $\Phi_{\gamma^{-1}}$ are linear independent, hence they form a basis of $V_{\mu(\gamma)}^+$ by Lemma 3.1(i). \square

It follows from Lemma 4.3 that $\phi_\gamma|_{I_+}$ and $\psi_\gamma|_{I_+}$ can be written uniquely as linear combination of Φ_γ and $\Phi_{\gamma^{-1}}$ for $\gamma \in S_{reg}$. The corresponding coefficients can be expressed in terms of the c -function $c(\gamma) = c(\gamma; a, b, c; z)$, which is defined by

$$c(\gamma) = \frac{1}{(ab, ac; q)_\infty \theta(-bcz)} \frac{(a/\gamma, b/\gamma, c/\gamma; q)_\infty \theta(-q/abcz\gamma)}{(1/\gamma^2; q)_\infty}. \quad (4.7)$$

Proposition 4.4. *Let $\gamma \in S_{reg}$. Then we have*

$$\phi_\gamma(x) = c(\gamma)\Phi_\gamma(x) + c(\gamma^{-1})\Phi_{\gamma^{-1}}(x), \quad x \in I_+.$$

The same formula holds for $\psi_\gamma(x)$ when $(a, b, c) \in V_z^{gen}$, with $c(\gamma)$ replaced by

$$\begin{aligned} \tilde{c}(\gamma; a, b, c; z) &= c(\gamma; a, q/b, q/c; bc z/q) \\ &= \frac{1}{(qa/b, qa/c; q)_\infty \theta(-1/z)} \frac{(a/\gamma, q/b\gamma, q/c\gamma; q)_\infty \theta(-1/az\gamma)}{(1/\gamma^2; q)_\infty}. \end{aligned} \quad (4.8)$$

Proof. We first prove the connection coefficient formula for ϕ_γ . Observe that $c(\gamma^{\pm 1})$ is well defined for $\gamma \in S_{reg}$ since $(a, b, c) \in V$. We fix $x = zq^k$ with $k \in \mathbb{Z}$ such that $q/acx < 1$. Furthermore, we assume that $a < 1$ and we fix $\gamma \in \mathbb{C} \setminus (-\infty, 0]$ such that $\gamma \in S_{reg}$ and $a < |\gamma| < 1/a$.

By the assumptions on x and γ , we may apply the three term recurrence relation [7, (3.3.3)] with $a \rightarrow -bcx$, $b \rightarrow b/\gamma$, $c \rightarrow c/\gamma$, $d \rightarrow -abcx/\gamma$ and $e \rightarrow bc$. We arrive at

$$\begin{aligned} & {}_3\phi_2 \left(\begin{matrix} -bcx, b/\gamma, c/\gamma \\ bc, -abcx/\gamma \end{matrix} ; q, a\gamma \right) \\ &= \frac{(c\gamma, b\gamma, -q/bx\gamma, -q\gamma/abcx; q)_\infty}{(bc, -q/abx, -q/bcx, \gamma^2; q)_\infty} {}_3\phi_2 \left(\begin{matrix} c/\gamma, a/\gamma, q/b\gamma \\ -q/bx\gamma, q/\gamma^2 \end{matrix} ; q, -q/acx \right) \\ &\quad - \frac{(-q\gamma/abcx, -q\gamma/ax, b/\gamma, c/\gamma, a/\gamma; q)_\infty \theta(-abcx\gamma/q)}{(-abcx/q\gamma, bc, -q/acx, -q/abx, -q/bcx; q)_\infty \theta(\gamma^2)} \\ &\quad \cdot {}_3\phi_2 \left(\begin{matrix} q\gamma/a, -q/acx, -q/abx \\ -q^2\gamma/abcx, -q\gamma/ax \end{matrix} ; q, a\gamma \right). \end{aligned}$$

The ${}_3\phi_2$ on the left hand side is the ${}_3\phi_2$ in the expression (3.8) for ϕ_γ and the second ${}_3\phi_2$ on the right hand side is the ${}_3\phi_2$ in the expression (4.2) for Φ_γ . Again by the assumptions on x and γ , we may rewrite the first ${}_3\phi_2$ on the right hand side using [7, (3.2.10)] with $a \rightarrow c/\gamma$, $b \rightarrow a/\gamma$, $c \rightarrow q/b\gamma$, $d \rightarrow q/\gamma^2$ and $e \rightarrow -q/bx\gamma$. This gives

$$\begin{aligned} & {}_3\phi_2 \left(\begin{matrix} c/\gamma, a/\gamma, q/b\gamma \\ -q/bx\gamma, q/\gamma^2 \end{matrix} ; q, -q/acx \right) \\ &= \frac{(a/\gamma, -q^2/abcx\gamma, -q/ax\gamma; q)_\infty}{(q/\gamma^2, -q/bx\gamma, -q/acx; q)_\infty} {}_3\phi_2 \left(\begin{matrix} q/a\gamma, -q/abx, -q/acx \\ -q^2/abcx\gamma, -q/ax\gamma \end{matrix} ; q, a/\gamma \right). \end{aligned}$$

The ${}_3\phi_2$ on the right hand side is the ${}_3\phi_2$ in the expression (4.2) for $\Phi_{\gamma^{-1}}$. So substituting this formula in the three term recurrence relation, and simplifying the formulas using in particular the functional relation

$$\theta(q^k x) = \begin{cases} q^{-k(k-1)/2} (-x)^{-k} \theta(x), & k \in \mathbb{Z}_+, \\ q^{k(k+1)/2} (-x)^{-k} \theta(x), & k \in -\mathbb{Z}_+ \end{cases} \quad (4.9)$$

for the Jacobi theta function, we arrive at the desired result for restricted choices of x , a and γ . The extension to all $x \in I_+$ is made using the fact that the left hand side and the right hand side of the c -function expansion are solutions of the eigenvalue equation $L_+ f = \mu(\gamma) f$ on I_+ . Finally, the restrictions on a and γ can be removed by analytic continuation.

The proof of the connection coefficient formula for ψ_γ follows by analytic continuation from the connection coefficient formula for ϕ_γ using (3.10) and (4.6). \square

For $(a, b, c) \in V_z^{gen}$ and $\gamma \in S_{reg}^+$ with $|\gamma| < a^{-1}$ we define $\Phi_\gamma^-(x)$ for $x \in (\infty(-1), 0]_q$ by

$$\begin{aligned} \Phi_\gamma^-(x) &= \frac{(-q^2\gamma/abcx, -q\gamma/ax, a\gamma; q)_\infty}{(-q/abx, -q/acx, q\gamma^2; q)_\infty} \\ &\quad \cdot (a\gamma)^{-k} {}_3\phi_2 \left(\begin{matrix} q\gamma/a, -q/abx, -q/acx \\ -q^2\gamma/abcx, -q\gamma/ax \end{matrix}; q, a\gamma \right), \quad x = -q^k. \end{aligned} \quad (4.10)$$

Observe that Φ_γ^- is obtained by taking $y = -1$ in the definition of the eigenfunction Φ_γ^y , see (4.2).

Lemma 4.5. *Let $(a, b, c) \in V_z^{gen}$ and $\gamma \in S_{reg}^+$ with $|\gamma| < a^{-1}$. Then*

$$\Phi_\gamma^-(x) = \frac{(q\gamma/a, q\gamma/b, q\gamma/c; q)_\infty}{(q/ab, q/ac, q\gamma^2; q)_\infty} \phi_\gamma(x), \quad x \in I_-.$$

Proof. Using the minimal solution [9, (2.26)] of the three term recurrence relation [9, (2.1)], it follows that Φ_γ^- is a solution of $Lf = \mu(\gamma)f$ on $(\infty(-1), 0]_q$, where L is the second order q -difference operator defined by (2.2). In particular, we have $A(-1)(\Phi_\gamma^-(-q) - \Phi_\gamma^-(-1)) = \mu(\gamma)\Phi_\gamma^-(-1)$ since $B(-1) = 0$. It follows that $\Phi_\gamma^-|_{I_-} \in V_{\mu(\gamma)}^-$.

We have $\dim(V_{\mu(\gamma)}^-) = 1$ by Lemma 3.1(i) and $0 \neq \phi_\gamma|_{I_-} \in V_{\mu(\gamma)}^-$. It follows that $\Phi_\gamma^-|_{I_-} = C_\gamma \phi_\gamma|_{I_-}$ for a unique constant C_γ . The explicit expression for C_γ can be found using $\phi_\gamma(-1) = 1$ and by evaluating $\Phi_\gamma^-(-1)$ using the q -Gauss sum [7, (1.5.1)]. \square

Observe that the Wronskian $W(\phi_\gamma, \Phi_\gamma^-)$ is identically zero on I_- by Lemma 4.5. On the other hand, the Wronskian $W(\phi_\gamma, \Phi_\gamma)$ on I_+ is non-zero for generic γ by Lemma 4.3 and Proposition 4.4. Suppose now that for a given (generic) $\gamma \in \mathbb{C}^*$ with $|\gamma| < a^{-1}$ we have an extension of $\Phi_\gamma \in V_{\mu(\gamma)}^+$ to a function $\Phi_\gamma \in V_{\mu(\gamma)}$. Then the Wronskian $W(\phi_\gamma, \Phi_\gamma)$ is constant on I by Lemma 3.1(iii) and Lemma 3.5, so $\Phi_\gamma|_{I_-}$ is *not* a multiple of Φ_γ^- on I_- .

This shows that the extension of $\Phi_\gamma \in V_{\mu(\gamma)}^+$ to a function $\Phi_\gamma \in V_{\mu(\gamma)}$ can not be obtained by taking the explicit expression for Φ_γ and extending its definition in an obvious manner to the whole q -interval I . We derive in the next section the extension of $\Phi_\gamma \in V_{\mu(\gamma)}^+$ to a function $\Phi_\gamma \in V_{\mu(\gamma)}$ for $(a, b, c) \in V_z^{gen}$ by writing Φ_γ explicitly as a linear combination of the two eigenfunctions $\phi_\gamma \in V_{\mu(\gamma)}$ and $\psi_\gamma \in V_{\mu(\gamma)}$.

5. THE EXTENSION OF THE ASYMPTOTIC SOLUTION

In this section we first evaluate the Wronskian $W(\psi_\gamma, \phi_\gamma)$ explicitly in product form. From this evaluation we derive explicit conditions on the spectral parameter γ for which we have $\dim(V_{\mu(\gamma)}) = 2$. For these values of the spectral parameter, there exists a unique extension of Φ_γ which lives in $V_{\mu(\gamma)}$ by Corollary 3.3. We will give this extension explicitly as a linear combination of ϕ_γ and ψ_γ .

Observe that if we compute the Wronskian by taking the limit to 0,

$$W(\psi_\gamma, \phi_\gamma)(0) = \frac{(1-q)^2}{bc} (\psi'_\gamma(0)\phi_\gamma(0) - \psi_\gamma(0)\phi'_\gamma(0)),$$

and by applying Lemma 3.5, we get an explicit expression as a linear combination of products of ${}_2\phi_2$'s, while evaluation of $W(\psi_\gamma, \phi_\gamma)(x)$ at $x = -1$ using (3.12) and $\phi_\gamma(-1) = 1$ gives an expression of the Wronskian as a linear combination of two ${}_3\phi_2$'s. From both these expressions, the (non)vanishing properties of the Wronskian $W(\psi_\gamma, \phi_\gamma)$ are hard to derive. In the next proposition we evaluate the Wronskian by substitution of the c -function expansions for ϕ_γ and ψ_γ .

Recall that the Wronskian $W(\psi_\gamma, \phi_\gamma)$ is constant on I by Lemma 3.1(iii) and Lemma 3.5.

Proposition 5.1. *Let $\gamma \in \mathbb{C}^*$ and $(a, b, c) \in V_z^{gen}$, then*

$$W(\psi_\gamma, \phi_\gamma) = (1-q) \frac{(a\gamma, a/\gamma; q)_\infty \theta(bc)}{(ab, ac, qa/b, qa/c; q)_\infty}. \quad (5.1)$$

Proof. We first assume that $\gamma \in S_{reg}$. Using Proposition 4.4 we then have

$$W(\psi_\gamma, \phi_\gamma) = \tilde{c}(\gamma)c(\gamma^{-1})W(\Phi_\gamma, \Phi_{\gamma^{-1}})(z) + \tilde{c}(\gamma^{-1})c(\gamma)W(\Phi_{\gamma^{-1}}, \Phi_\gamma)(z), \quad (5.2)$$

where $c(\cdot)$ and $\tilde{c}(\cdot)$ are given by (4.7) and (4.8), respectively. By Lemma 4.3, the right hand side of (5.2) is equal to

$$W(\psi_\gamma, \phi_\gamma) = -aK (c(\gamma)\tilde{c}(\gamma^{-1}) - c(\gamma^{-1})\tilde{c}(\gamma)) (\gamma - \gamma^{-1}), \quad (5.3)$$

where K is the positive constant defined by (2.14). It follows by direct computation that

$$\begin{aligned} c(\gamma)\tilde{c}(\gamma^{-1}) - c(\gamma^{-1})\tilde{c}(\gamma) &= \frac{1}{(ab, ac, qa/b, qa/c; q)_\infty \theta(-q/bcz, -1/z)} \\ &\cdot \frac{(a\gamma, a/\gamma; q)_\infty}{(\gamma^2, \gamma^{-2}; q)_\infty} \left\{ \theta(q\gamma/b, c/\gamma, -q/abcz\gamma, -\gamma/az) \right. \\ &\quad \left. - \theta(q/b\gamma, c\gamma, -q\gamma/abcz, -1/az\gamma) \right\}. \end{aligned}$$

Now we can apply the θ -product identity

$$\theta(x\lambda, x/\lambda, \mu\nu, \mu/\nu) - \theta(x\nu, x/\nu, \lambda\mu, \mu/\lambda) = \frac{\mu}{\lambda} \theta(x\mu, x/\mu, \lambda\nu, \lambda/\nu), \quad (5.4)$$

see [7, Exercise 2.16], with parameter values

$$x = c\sqrt{\frac{q}{bc}}, \quad \lambda = \gamma\sqrt{\frac{q}{bc}}, \quad \mu = -\frac{1}{az}\sqrt{\frac{q}{bc}}, \quad \nu = \frac{1}{\gamma}\sqrt{\frac{q}{bc}},$$

to obtain

$$\begin{aligned} c(\gamma)\tilde{c}(\gamma^{-1}) - c(\gamma^{-1})\tilde{c}(\gamma) &= \frac{-1}{(ab, ac, qa/b, qa/c; q)_\infty \theta(-q/bcz, -1/z)} \\ &\cdot \frac{1}{az\gamma} \frac{(a\gamma, a/\gamma; q)_\infty}{(\gamma^2, \gamma^{-2}; q)_\infty} \theta(-q/abz, -acz, q/bc, \gamma^2). \end{aligned} \quad (5.5)$$

The proposition now follows for $\gamma \in S_{reg}$ by substitution of (5.5) in (5.3) and using $\theta(x) = \theta(q/x)$. By continuity in γ , it follows that (5.1) holds for all $\gamma \in \mathbb{C}^*$. \square

In the following corollary we give a characterization of the eigenfunction ϕ_γ of L for generic values of the spectral parameter γ . In section 9 we show that the eigenfunction ϕ_γ plays the role of the spherical function for the big q -Jacobi function transform.

Corollary 5.2. *Let $\gamma \in \mathbb{C}^*$ and $(a, b, c) \in V_z^{gen}$.*

(i) *$\{\phi_\gamma, \psi_\gamma\}$ is a linear basis of $V_{\mu(\gamma)}$ if*

$$\gamma \notin \{aq^n, a^{-1}q^{-n}\}_{n \in \mathbb{Z}_+}. \quad (5.6)$$

(ii) *Assume that (5.6) is satisfied. Let $f \in \mathcal{F}(I)$ be a function satisfying*

- $Lf = \mu(\gamma)f$ on I ;
- f is continuously differentiable at the origin;
- $f(-1) = 1$.

Then $f = \phi_\gamma$.

(iii) *If (5.6) is satisfied, then $(L\psi_\gamma)(-1) \neq \mu(\gamma)\psi_\gamma(-1)$.*

Proof. We fix $\gamma \in \mathbb{C}^*$ satisfying (5.6). Then the Wronskian $W(\psi_\gamma, \phi_\gamma)$ is non-zero since $(a, b, c) \in V_z^{gen}$, see Proposition 5.1. Hence $\{\phi_\gamma, \psi_\gamma\}$ is a linear basis of $V_{\mu(\gamma)}$ by Proposition 3.2 and Lemma 3.5.

For the proof of (ii), we observe first that ϕ_γ satisfies the three properties as stated in (ii), see section 3. If f is another function satisfying the same three properties, then $f|_{I_-} = \phi_\gamma|_{I_-}$ since $f(-1) = \phi_\gamma(-1) = 1$ and $\dim(V_{\mu(\gamma)}^-) = 1$ by Lemma 3.1. Hence $f = \phi_\gamma$ on I by Corollary 3.3.

Finally, for the proof of (iii), observe that $\psi_\gamma|_{I_-}$ and $\phi_\gamma|_{I_-}$ are linearly independent by part (i) of the corollary and by Proposition 3.2. Since $\phi_\gamma|_{I_-} \in V_{\mu(\gamma)}^-$ by part (ii) of the corollary and $\dim(V_{\mu(\gamma)}^-) = 1$ by Lemma 3.1(i), it follows that $\psi_\gamma|_{I_-} \in \hat{V}_{\mu(\gamma)}^- \setminus V_{\mu(\gamma)}^-$. So $(L\psi_\gamma)(-1) \neq \mu(\gamma)\psi_\gamma(-1)$, as desired. \square

Set $\gamma_n = aq^n$ ($n \in \mathbb{Z}_+$). We have the following description of the excluded set $\{\gamma_n^{\pm 1}\}_{n \in \mathbb{Z}_+}$ in Corollary 5.2.

Proposition 5.3. *Let $S_{pol} \subset \mathbb{C}^*$ be the set of spectral parameters $\gamma \in \mathbb{C}^*$ for which the eigenvalue equation $(Lf)(x) = \mu(\gamma)f(x)$ on I has a solution $f \neq 0$ which is polynomial in x . Then $S_{pol} = \{\gamma_n^{\pm 1}\}_{n \in \mathbb{Z}_+}$. The non-zero polynomial eigenfunction corresponding to the eigenvalue $\mu(\gamma_n^{\pm 1})$ is the big q -Jacobi polynomial of degree n and is explicitly given by*

$$\begin{aligned} \phi_{\gamma_n}(x) &= \frac{(qa/b, qa/c; q)_n}{(ab, ac; q)_n} \left(\frac{bc}{q}\right)^n \psi_{\gamma_n}(x) \\ &= \frac{(qa/c; q)_n}{(ac; q)_n} \left(\frac{-c}{aq^{(n+1)/2}}\right)^n {}_3\phi_2 \left(\begin{matrix} q^{-n}, -abx, q^n a^2 \\ ab, qa/c \end{matrix}; q, q \right). \end{aligned} \quad (5.7)$$

Proof. This is well known, see for instance [1] and [7, section 7.3]. The connection between ϕ_{γ_n} and ψ_{γ_n} with the ${}_3\phi_2$ in the last equality of (5.7) follows from [7, (3.2.5)]. \square

Remark 5.4. Proposition 5.1 and Proposition 5.3 are essential ingredients for a functional-analytic derivation of the orthogonality relations and the quadratic norm evaluations for the big q -Jacobi polynomials, see section 10.

We write $S_{pol}^\pm = \{\gamma_n^{\pm 1}\}_{n \in \mathbb{Z}_+}$, where $\gamma_n = aq^n$. Observe that $S_{pol}^+ \subset S_{pol} \subset S_{reg} \subset S_{reg}^+$ when $(a, b, c) \in V_z^{gen}$.

Proposition 5.5. *Let $\gamma \in S_{reg}^+ \setminus S_{pol}^+$ and $(a, b, c) \in V_z^{gen}$. Then*

$$\Phi_\gamma(x) = K(\gamma)\phi_\gamma(x) + \tilde{K}(\gamma)\psi_\gamma(x), \quad x \in I_+, \quad (5.8)$$

with $K(\gamma) = K(\gamma; a, b, c; z)$ given by

$$K(\gamma) = \frac{(ab, ac, q\gamma/b, q\gamma/c; q)_\infty}{(q\gamma^2, a/\gamma; q)_\infty} \frac{\theta(-bcz, -az/\gamma)}{\theta(bc, -abz, -acz)},$$

and $\tilde{K}(\gamma) = \tilde{K}(\gamma; a, b, c; z) = K(\gamma; a, q/b, q/c; bcz/q)$.

Proof. We first prove (5.8) for $\gamma \in S_{reg} \setminus S_{pol}$. By Corollary 5.2(i) and Proposition 3.2, there exist unique $K(\gamma), \tilde{K}(\gamma) \in \mathbb{C}$ such that (5.8) holds for all $x \in I_+$. These coefficients can be expressed in terms of Wronskians by

$$K(\gamma) = \frac{W(\psi_\gamma, \Phi_\gamma)(z)}{W(\psi_\gamma, \phi_\gamma)}, \quad \tilde{K}(\gamma) = \frac{W(\Phi_\gamma, \phi_\gamma)(z)}{W(\psi_\gamma, \phi_\gamma)}. \quad (5.9)$$

By Lemma 4.3 and Proposition 4.4 we have

$$\begin{aligned} W(\psi_\gamma, \Phi_\gamma)(z) &= aK\tilde{c}(\gamma^{-1})(\gamma^{-1} - \gamma), \\ W(\Phi_\gamma, \phi_\gamma)(z) &= aKc(\gamma^{-1})(\gamma - \gamma^{-1}), \end{aligned} \quad (5.10)$$

where K is the positive constant defined by (2.14). Formula (5.8) now follows by substituting (5.10), the explicit formula (5.1) for the Wronskian $W(\psi_\gamma, \phi_\gamma)$, and the explicit expressions (4.7) and (4.8) for $c(\gamma)$ and $\tilde{c}(\gamma)$ in (5.9), and using the theta function identities $\theta(x) = \theta(q/x)$ and (4.9).

It follows now by continuity in γ that (5.8) is valid for $\gamma \in S_{reg}^+ \setminus S_{pol}^+$. \square

Remark 5.6. Fix $\gamma \in S_{reg} \setminus S_{pol}$ and suppose that $(a, b, c) \in V_z^{gen}$. The proof of Proposition 5.1 implies the matrix equation

$$\begin{pmatrix} c(\gamma) & c(\gamma^{-1}) \\ \tilde{c}(\gamma) & \tilde{c}(\gamma^{-1}) \end{pmatrix} = \begin{pmatrix} K(\gamma) & \tilde{K}(\gamma) \\ K(\gamma^{-1}) & \tilde{K}(\gamma^{-1}) \end{pmatrix}^{-1}. \quad (5.11)$$

Indeed, (5.11) follows easily from the explicit formula (5.5) for the determinant of the matrix on the left hand side of (5.11). Let $x \in I_+$, then the matrix equation (5.11) implies that the two connection coefficient formulas

$$\phi_\gamma(x) = c(\gamma)\Phi_\gamma(x) + c(\gamma^{-1})\Phi_{\gamma^{-1}}(x), \quad \psi_\gamma(x) = \tilde{c}(\gamma)\Phi_\gamma(x) + \tilde{c}(\gamma^{-1})\Phi_{\gamma^{-1}}(x)$$

are equivalent to the two connection coefficient formulas $\Phi_{\gamma^{\pm 1}}(x) = K(\gamma^{\pm 1})\phi_\gamma(x) + \tilde{K}(\gamma^{\pm 1})\psi_\gamma(x)$.

Corollary 5.7. *Let $\gamma \in S_{reg}^+ \setminus S_{pol}$ and $(a, b, c) \in V_z^{gen}$. The unique extension of $\Phi_\gamma \in V_{\mu(\gamma)}^+$ to a function $\Phi_\gamma \in V_{\mu(\gamma)}$ is given by*

$$\Phi_\gamma(x) = K(\gamma)\phi_\gamma(x) + \tilde{K}(\gamma)\psi_\gamma(x), \quad x \in I, \quad (5.12)$$

with $K(\gamma), \tilde{K}(\gamma)$ as defined in Proposition 5.1.

Proof. Immediate from Corollary 3.3, Corollary 5.2(i) and Proposition 5.5. \square

In section 4 we have seen that $\Phi_\gamma(x)$ depends analytically on $\gamma \in S_{reg}^+$ for all $x \in I_+$. We have the following analogous result for the extension (5.12) of $\Phi_\gamma \in V_{\mu(\gamma)}^+$.

Lemma 5.8. *Let $\gamma \in S_{pol}$ and $(a, b, c) \in V_z^{gen}$. There exists a unique extension of $\Phi_\gamma \in V_{\mu(\gamma)}^+$ to a function $\Phi_\gamma \in V_{\mu(\gamma)}$ such that $\tilde{\gamma} \mapsto \Phi_{\tilde{\gamma}}(x)$ is analytic at $\tilde{\gamma} = \gamma$ for all $x \in I$.*

If $\gamma = \gamma_n^{-1} \in S_{pol}^-$, then the extension of Φ_γ is given by (5.12). If $\gamma = \gamma_n \in S_{pol}^+$, then the extension of Φ_{γ_n} is given by

$$\Phi_{\gamma_n}(x) = \left(\text{Res}_{\gamma=\gamma_n} K(\gamma) \right) \frac{\partial \phi_\gamma}{\partial \gamma}(x)|_{\gamma=\gamma_n} + M_n \phi_{\gamma_n}(x) + \left(\text{Res}_{\gamma=\gamma_n} \tilde{K}(\gamma) \right) \frac{\partial \psi_\gamma}{\partial \gamma}(x)|_{\gamma=\gamma_n} \quad (5.13)$$

for $x \in I$, with M_n given by the existing limit $M_n = \lim_{\gamma \rightarrow \gamma_n} M_n(\gamma)$, where

$$M_n(\gamma) = \frac{(qa/b, qa/c; q)_n}{(ab, ac; q)_n} \left(\frac{bc}{q} \right)^n K(\gamma) + \tilde{K}(\gamma). \quad (5.14)$$

Proof. The proof for $\gamma \in S_{pol}^-$ is trivial since $K(\tilde{\gamma})$, $\tilde{K}(\tilde{\gamma})$, $\phi_{\tilde{\gamma}}(x)$ and $\psi_{\tilde{\gamma}}(x)$ are regular at $\tilde{\gamma} = \gamma$.

For $\gamma_n \in S_{pol}^+$ ($n \in \mathbb{Z}_+$), observe that $K(\gamma)$ and $\tilde{K}(\gamma)$ have simple poles at $\gamma = \gamma_n$ and that $\phi_\gamma(x)$ and $\psi_\gamma(x)$ are regular at $\gamma = \gamma_n$. It follows from (5.12) and the first equality of (5.7) that the singularity of $\Phi_\gamma(x)$ at $\gamma = \gamma_n$ is removable for $x \in I$ if the (at most simple) singularity of $M_n(\gamma)$ at $\gamma = \gamma_n$ is removable. This can be checked by direct computations using the theta function identities $\theta(x) = \theta(q/x)$ and (4.9).

It follows that $\lim_{\gamma \rightarrow \gamma_n} M_n(\gamma)$ exists and that (5.13) holds. The extension (5.13) of Φ_{γ_n} lies in $V_{\mu(\gamma_n)}$ because ϕ_{γ_n} and the derivatives of ϕ_γ and ψ_γ at $\gamma = \gamma_n$ are continuously differentiable at the origin, cf. Lemma 3.5. \square

For future reference, we collect here the main results concerning the asymptotic solution of the eigenvalue equation $Lf = \mu(\gamma)f$.

Theorem 5.9. *Let $x \in I$ and $(a, b, c) \in V_z^{gen}$.*

- (i) $\Phi_\gamma \in V_{\mu(\gamma)}$ for all $\gamma \in S_{reg}^+$.
- (ii) $\Phi_\gamma(x)$ is analytic at $\gamma \in S_{reg}^+$.
- (iii) $\Phi_\gamma \in \mathcal{D}$ for $\gamma \in \mathbb{C}^*$ with $|\gamma| < 1$.
- (iv) $\phi_\gamma(x) = c(\gamma)\Phi_\gamma(x) + c(\gamma^{-1})\Phi_{\gamma^{-1}}(x)$ for all $\gamma \in S_{reg}$.
- (v) Let $\gamma \in S_{reg}^+$. The Wronskian $W(\gamma) = W(\Phi_\gamma, \phi_\gamma)$ is constant on I , and $W(\gamma) = aKc(\gamma^{-1})(\gamma - \gamma^{-1})$, where K is the positive constant defined by (2.14).

Proof. (i) and (ii) follow from Corollary 5.7 and Lemma 5.8.

(iii) Let $\gamma \in \mathbb{C}^*$ with $|\gamma| < 1$, then $\gamma \in S_{reg}^+$, hence Φ_γ is well defined. Then $\Phi_\gamma \in \mathcal{D}$ follows from part (i) of the theorem and from Lemma 4.2.

(iv) We first prove the connection coefficient formula for $\gamma \in S_{reg} \setminus S_{pol}$. Then the connection coefficient formula is valid for all $x \in I_+$, see Proposition 4.4. Since $\phi_\gamma, \Phi_{\gamma^{\pm 1}} \in V_{\mu(\gamma)}$, it follows by the uniqueness property of extensions of eigenfunctions, see Corollary 3.3 and Corollary 5.2(i), that the connection coefficient formula is valid for all $x \in I$. The connection coefficient formula holds then for all $\gamma \in S_{reg}$ by continuity.

(v) This follows from Lemma 3.1(iii), part (iv) of the theorem, and Lemma 4.3. \square

Remark 5.10. Observe that the statements of Theorem 5.9(i) and (iii) are not in contradiction with the self-adjointness of (L, \mathcal{D}) , see Proposition 2.7. Indeed, let $\gamma \in \mathbb{C} \setminus \mathbb{R}$ with $|\gamma| < 1$, then $\gamma \in S_{reg}^+$ and $\mu(\gamma) \in \mathbb{C} \setminus \mathbb{R}$. By Theorem 5.9(i) and (iii) we have that $\Phi_\gamma \in \mathcal{D}$ and $(L\Phi_\gamma)(x) = \mu(\gamma)\Phi_\gamma(x)$ for all $x \in (-1, \infty(z))_q$, but the eigenvalue equation does not hold in the end-point $x = -1$. In fact, the self-adjointness of (L, \mathcal{D}) forces that $(L\Phi_\gamma)(-1) \neq \mu(\gamma)\Phi_\gamma(-1)$. Another proof of this inequality can be given using the non-vanishing of the Wronskian $W(\phi_\gamma, \Phi_\gamma)$, cf. the proof of Corollary 5.2(iii).

6. THE GREEN FUNCTION

In this section we define the Green function of the self-adjoint operator (L, \mathcal{D}) . With the proper extensions of the asymptotic expansion Φ_γ now at hand, see Theorem 5.9, the construction of the Green function is a straightforward extension of the construction given by Kakehi [11] and Kakehi, Masuda and Ueno [12] for the little q -Jacobi function transform. We use some standard terminology and results for unbounded self-adjoint operators and their spectral measures for which we refer to Dunford and Schwartz [6, Chapter XII] and Rudin [20, Chapter 13].

Let $(a, b, c) \in V_z^{gen}$ and $\mu \in \mathbb{C} \setminus \mathbb{R}$. Let $\gamma \in \mathbb{C}^*$ be the unique non-zero complex number such that $\mu = \mu(\gamma)$ and $|\gamma| < 1$. Observe that $\gamma \in \mathbb{C} \setminus \mathbb{R}$, hence $\gamma \in S_{reg}$ and $W(\gamma) \neq 0$, see Theorem 5.9(v). We define the Green kernel $K_\gamma(x, y)$ for $x, y \in I$ by

$$K_\gamma(x, y) = \begin{cases} W(\gamma)^{-1} \Phi_\gamma(x) \phi_\gamma(y), & y \leq x, \\ W(\gamma)^{-1} \phi_\gamma(x) \Phi_\gamma(y), & y \geq x. \end{cases} \quad (6.1)$$

Observe that $K_\gamma(x, \cdot), K_\gamma(\cdot, x) \in \mathcal{D}$ for all $x \in I$ in view of Lemma 3.5 and Theorem 5.9(i) and (iii), so we have a well defined linear map $\mathcal{H} \rightarrow \mathcal{F}(I)$ mapping $f \in \mathcal{H}$ to

$$G_f(x, \gamma) = \langle f, \overline{K_\gamma(x, \cdot)} \rangle, \quad x \in I. \quad (6.2)$$

Written out explicitly, we arrive at the formula

$$G_f(x, \gamma) = W(\gamma)^{-1} \left(\Phi_\gamma(x) \int_{-1}^x f(y) \phi_\gamma(y) \frac{d_q y}{p(y)} + \phi_\gamma(x) \int_x^{\infty(z)} f(y) \Phi_\gamma(y) \frac{d_q y}{p(y)} \right). \quad (6.3)$$

By the self-adjointness of (L, \mathcal{D}) we have that the resolvent $(L - \mu \cdot \text{Id})^{-1}$ is a one to one, continuous map from \mathcal{H} onto \mathcal{D} for all $\mu \in \mathbb{C} \setminus \mathbb{R}$.

Proposition 6.1. *Let $(a, b, c) \in V_z^{gen}$, $f \in \mathcal{H}$ and $\mu = \mu(\gamma) \in \mathbb{C} \setminus \mathbb{R}$ with $\gamma \in \mathbb{C}^*$ and $|\gamma| < 1$. Then $G_f(\cdot, \gamma) = (L - \mu \cdot \text{Id})^{-1} f$. In particular, $G_f(\cdot, \gamma) \in \mathcal{D}$.*

Proof. We first prove that

$$((L - \mu(\gamma))G_f(\cdot, \gamma))(x) = f(x), \quad \forall f \in \mathcal{H}. \quad (6.4)$$

For the proof we need to consider the two cases $x \in (-1, \infty(z))_q$ and $x = -1$ separately.

Case 1: $x \in (-1, \infty(z))_q$.

Observe that the product rule (3.3) for the q -derivative D_q and Lemma 2.1 imply the following product rule for L :

$$\begin{aligned} (L(fg))(x) &= (Lf)(x)g(x) + f(x)(Lg)(x) + qp(x)r(x)(D_q f)(x)(D_q g)(x) \\ &\quad + p(x)r(q^{-1}x)(D_q f)(q^{-1}x)(D_q g)(q^{-1}x). \end{aligned}$$

Combined with the easily verified formulas

$$D_q \left(x \mapsto \int_{-1}^x f(y) d_q y \right) = f(x), \quad D_q \left(x \mapsto \int_x^{\infty(z)} f(y) d_q y \right) = -f(x)$$

and the definition (6.3) of $G_f(x, \gamma)$, we obtain

$$\begin{aligned} W(\gamma) \left((L - \mu(\gamma)) G_f(\cdot, \gamma) \right) (x) &= \\ &\Phi_\gamma(x)p(x) (D_q(p^{-1}rf\phi_\gamma)) (q^{-1}x) - \phi_\gamma(x)p(x) (D_q(p^{-1}rf\Phi_\gamma)) (q^{-1}x) \\ &\quad + p(x) (p^{-1}rf(D_q\Phi_\gamma)\phi_\gamma) (q^{-1}x) - p(x) (p^{-1}rf(D_q\phi_\gamma)\Phi_\gamma) (q^{-1}x) \\ &\quad + q (rf(D_q\Phi_\gamma)\phi_\gamma) (x) - q (rf(D_q\phi_\gamma)\Phi_\gamma) (x) \end{aligned}$$

for $f \in \mathcal{H}$. By writing out the D_q -terms in this formula, we see that the right hand side reduces to $W(\Phi_\gamma, \phi_\gamma)(x)f(x)$, which in turn is equal to $W(\gamma)f(x)$ by Theorem 5.9(v). This completes the proof of (6.4) for $x \in (-1, \infty(z))_q$.

Case 2: $x = -1$.

Observe that the Green function in the end-point -1 reduces to

$$G_f(-1, \gamma) = \int_{-1}^{\infty(z)} f(y) \Phi_\gamma(y) \frac{d_q y}{p(y)}$$

since $\phi_\gamma(-1) = 1$. Using the product rule (3.3), we then have that

$$\begin{aligned} W(\gamma) (D_q G_f(\cdot, \gamma)) (-1) &= \Phi_\gamma(-q) \phi_\gamma(-1) \frac{f(-1)}{p(-1)} \\ &\quad - \phi_\gamma(-q) \Phi_\gamma(-1) \frac{f(-1)}{p(-1)} + (D_q \phi_\gamma)(-1) G_f(-1, \gamma). \end{aligned}$$

Combined with the expression of L in the end-point $x = -1$ in terms of the q -derivative and the functions $p(\cdot)$ and $r(\cdot)$, see Lemma 2.1, and using the fact that $(L\phi_\gamma)(-1) = \mu(\gamma)\phi_\gamma(-1)$, we obtain

$$W(\gamma) \left((L - \mu(\gamma)) G_f(\cdot, \gamma) \right) (-1) = W(\Phi_\gamma, \phi_\gamma)(-1)f(-1).$$

Hence (6.4) is valid for the end-point $x = -1$ since $W(\Phi_\gamma, \phi_\gamma)(-1) = W(\gamma)$ by Theorem 5.9(v).

It remains to show how (6.4) leads to a proof of the proposition. Let \mathcal{D}_{fin} be the set of functions $f : I \rightarrow \mathbb{C}$ with finite support. Then $\mathcal{D}_{fin} \subset \mathcal{D} \subset \mathcal{H}$ as dense subspaces. Let $f \in \mathcal{D}_{fin}$, then $G_f(\cdot, \gamma)$ is continuously differentiable at the origin since $\phi_\gamma, \Phi_\gamma \in V_{\mu(\gamma)}$. Furthermore, $G_f(x, \gamma)$ is a constant multiple of $\Phi_\gamma(x)$ for $x \gg 0$, hence $G_f(\cdot, \gamma) \in \mathcal{H}$. By (6.4) it follows that $LG_f(\cdot, \gamma) \in \mathcal{H}$, hence $G_f(\cdot, \gamma) \in \mathcal{D}$. Combined with (6.4), this proves that

$$((L - \mu(\gamma) \cdot \text{Id})^{-1} f)(x) = G_f(x, \gamma) = \langle f, \overline{K_\gamma(x, \cdot)} \rangle, \quad \forall f \in \mathcal{D}_{fin} \quad (6.5)$$

for all $x \in I$. By continuity, (6.5) is valid for all $f \in \mathcal{H}$. This completes the proof of the proposition. \square

Remark 6.2. Proposition 6.1 is in general not valid when ϕ_γ is replaced by ψ_γ and $W(\gamma)$ is replaced by the constant value of $W(\Phi_\gamma, \psi_\gamma)$ on I in the definition of the Green function $G_f(\cdot, \gamma)$. Indeed, for the proof of $((L - \mu(\gamma))G_f(\cdot, \gamma))(-1) = f(-1)$ in the previous proposition, we use the fact that ϕ_γ is a solution of $(Lf)(x) = \mu(\gamma)f(x)$ in the end-point $x = -1$. This property fails in general to be true for ψ_γ , see Corollary 5.2(iii).

Proposition 6.1 plays a crucial role in determining the explicit form of the resolution of the identity $L = \int_{\mathbb{R}} t dE(t)$ for the self-adjoint operator (L, \mathcal{D}) on \mathcal{H} since the spectral measure E is related to the resolvent of L by

$$\begin{aligned} & \langle E((\mu_1, \mu_2))f, g \rangle \\ &= \lim_{\delta \downarrow 0} \lim_{\epsilon \downarrow 0} \frac{1}{2\pi i} \int_{\mu_1 + \delta}^{\mu_2 - \delta} \left(\langle (L - (\mu + i\epsilon))^{-1} f, g \rangle - \langle (L - (\mu - i\epsilon))^{-1} f, g \rangle \right) d\mu, \end{aligned} \quad (6.6)$$

where $\mu_1 < \mu_2$ and $f \in \mathcal{D}$, $g \in \mathcal{H}$, see [6, Theorem XII.2.10]. In the following two sections we use Proposition 6.1 and (6.6) to give an explicit description of the continuous and discrete contributions to the spectral resolution E .

7. THE CONTINUOUS SPECTRUM

We start this section by proving that the closed interval $[-(1+a)^2, -(1-a)^2]$ is contained in the continuous spectrum $\sigma_c(L)$ of (L, \mathcal{D}) . In section 8 we will see in fact that this interval is exactly equal to $\sigma_c(L)$ for $(a, b, c) \in V_z^{gen}$. Furthermore, we compute the spectral projection $P_c := E([- (1+a)^2, -(1-a)^2])$ explicitly, and give the Plancherel formula and inversion formula for the continuous part of the big q -Jacobi function transform.

For $n \in \mathbb{N}$ and $x \in I$ we set $\varphi_\gamma^{(n)}(x) = \phi_\gamma^{(n)}(x) / \|\phi_\gamma^{(n)}\|$, where $\phi_\gamma^{(n)} \in \mathcal{H}$ is defined by

$$\phi_\gamma^{(n)}(x) = \begin{cases} \phi_\gamma(x) & \text{if } x \in I \setminus [zq^{-n-1}, \infty(z))_q, \\ 0 & \text{if } x \in [zq^{-n-1}, \infty(z))_q. \end{cases}$$

Lemma 7.1. *Let $\mu \in [-(1+a)^2, -(1-a)^2]$ and $\theta \in [0, \pi]$ such that $\mu = \mu(e^{i\theta})$. Then $(\varphi_{e^{i\theta}}^{(n)})_{n \in \mathbb{N}}$ is a generalized eigenfunction of (L, \mathcal{D}) with generalized eigenvalue μ . Furthermore, the continuous spectrum $\sigma_c(L)$ of the self-adjoint operator (L, \mathcal{D}) contains the interval $[-(1+a)^2, -(1-a)^2]$.*

Proof. Recall that $(\varphi_{e^{i\theta}}^{(n)})_{n \in \mathbb{N}}$ is a generalized eigenfunction of (L, \mathcal{D}) with generalized eigenvalue $\mu(e^{i\theta})$ if $\varphi_{e^{i\theta}}^{(n)} \in \mathcal{D}$, $\|\varphi_{e^{i\theta}}^{(n)}\| = 1$ and $\lim_{n \rightarrow \infty} (L\varphi_{e^{i\theta}}^{(n)} - \mu(e^{i\theta})\varphi_{e^{i\theta}}^{(n)}) = 0$ in \mathcal{H} . This can be checked for all $\theta \in [0, \pi]$ by an elementary computation using Proposition 5.2(ii).

Let $\mu \in [-(1+a)^2, -(1-a)^2]$. Then μ is part of the spectrum $\sigma(L)$ of (L, \mathcal{D}) since there exists a generalized eigenfunction of (L, \mathcal{D}) with generalized eigenvalue μ . Then μ is in the continuous spectrum $\sigma_c(L)$ of (L, \mathcal{D}) or in the point spectrum $\sigma_p(L)$ of (L, \mathcal{D}) since a self-adjoint operator does not have residual spectrum, see [20, Theorem 13.27].

It remains to show that $\mu \notin \sigma_p(L)$. We have to distinguish between the cases $\mu \in (-(1+a)^2, -(1-a)^2)$ and $\mu = \mu(\pm 1) = -(1 \mp a)^2$.

Case 1: $\mu \in (-(1+a)^2, -(1-a)^2)$, i.e. $\mu = \mu(e^{i\theta})$ with $\theta \in (0, \pi)$.

It is a straightforward consequence of the asymptotic behaviour of Φ_γ to infinity, see (4.4), that any non-zero linear combination of the basis elements $\{\Phi_{e^{i\theta}}, \Phi_{e^{-i\theta}}\}$ of $V_{\mu(e^{i\theta})}^+$ does not lie in $M_{\mu(e^{i\theta})}$ (see Lemma 4.2 for the definition of M_μ), hence $\mu \notin \sigma_p(L)$.

Case 2: $\mu = \mu(\pm 1) = -(1 \mp a)^2$.

Observe that $\frac{\partial \Phi_\gamma}{\partial \gamma}|_{\gamma=\pm 1} \in V_{\mu(\pm 1)}^+$ since $\frac{\partial \mu}{\partial \gamma}(\gamma)|_{\gamma=\pm 1} = 0$. Using that the asymptotics to infinity of $\frac{\partial \Phi_\gamma}{\partial \gamma}$ is given by

$$\frac{\partial \Phi_\gamma}{\partial \gamma}(zq^{-m}) = m\gamma^{-1}(a\gamma)^m(1 + \mathcal{O}(q^m)), \quad m \rightarrow \infty,$$

we obtain that

$$W(\Phi_{\pm 1}, \frac{\partial \Phi_\gamma}{\partial \gamma}|_{\gamma=\pm 1})(x) = -aK \neq 0, \quad x \in I_+,$$

compare with the proof of Lemma 4.3. Combined with Lemma 3.1(i), it follows that $\{\Phi_{\pm 1}, \frac{\partial \Phi_\gamma}{\partial \gamma}|_{\gamma=\pm 1}\}$ is a basis of $V_{\mu(\pm 1)}^+$. It is easy to see, using asymptotics to infinity, that any non-zero linear combination of these basis elements does not lie in $M_{\mu(\pm 1)}$, hence $\mu(\pm 1) \notin \sigma_p(L)$. \square

Let $\mathcal{D}_{fin} \subset \mathcal{H}$ be the linear subspace of functions $f : I \rightarrow \mathbb{C}$ with finite support. Observe that $\mathcal{D}_{fin} \subset \mathcal{D} \subset \mathcal{H}$ as dense subspaces. We define the big q -Jacobi function transform by

$$(\mathcal{F}f)(\gamma) = \langle f, \phi_\gamma \rangle = \int_{-1}^{\infty(z)} f(x) \overline{\phi_\gamma(x)} \frac{d_q x}{p(x)}, \quad f \in \mathcal{D}_{fin}, \quad \gamma \in \mathbb{C}^*. \quad (7.1)$$

In this section, we will regard the big q -Jacobi function transform $\mathcal{F}f$ of $f \in \mathcal{D}_{fin}$ as a function on the unit circle \mathbb{T} . Observe that $\mathcal{F}f$ is W -invariant, where $W = \{\pm 1\}$ acts by $(\epsilon.g)(\gamma) = g(\gamma^\epsilon)$, $\epsilon \in W$.

We define an absolutely continuous measure on \mathbb{T} by

$$d\nu(\gamma) = \frac{1}{4\pi i K} \frac{d\gamma}{c(\gamma)c(\gamma^{-1})\gamma} = \frac{1}{4\pi i K} \frac{d\gamma}{|c(\gamma)|^2 \gamma}, \quad (7.2)$$

where K is the positive constant defined by (2.14). Observe that the measure $d\nu$ is a well defined measure on \mathbb{T} since $(a, b, c) \in V$. In particular, possible zeros of the denominator of the weight function $1/|c(\cdot)|^2$ are compensated by zeros of the numerator.

We first show that \mathcal{F} extends uniquely to a partial isometry $\mathcal{F} : \mathcal{H} \rightarrow L_W^2(\mathbb{T}, d\nu)$, where $L_W^2(\mathbb{T}, d\nu)$ is the L^2 -space of W -invariant functions with respect to $d\nu$, when $(a, b, c) \in V_z^{gen}$. We start with the following crucial consequence of (6.6) and Proposition 6.1.

Proposition 7.2. *Let $(a, b, c) \in V_z^{gen}$. Choose $-(1+a)^2 \leq \mu_1 < \mu_2 \leq -(1-a)^2$ and let $0 \leq \theta_2 < \theta_1 \leq \pi$ such that $\mu_j = \mu(e^{i\theta_j})$ ($j = 1, 2$). Then*

$$\langle E((\mu_1, \mu_2))f, g \rangle = \frac{1}{2\pi K} \int_{\theta_2}^{\theta_1} (\mathcal{F}f)(e^{i\theta}) \overline{(\mathcal{F}g)(e^{i\theta})} \frac{d\theta}{|c(e^{i\theta})|^2} \quad (7.3)$$

for all $f, g \in \mathcal{D}_{fin}$.

Proof. For $\mu \in \mathbb{C} \setminus \mathbb{R}$ we write $\gamma[\mu] \in \mathbb{C} \setminus \mathbb{R}$ for the unique complex number with modulus less than 1 such that $\mu(\gamma[\mu]) = \mu$. Fix $f, g \in \mathcal{D}_{fin}$. By a straightforward computation using Proposition 6.1, we have for $\mu \in \mathbb{R}$ and $\epsilon > 0$ that

$$\begin{aligned} \langle (L - (\mu \pm i\epsilon))^{-1} f, g \rangle &= \iint_{\substack{(x,y) \in I \times I \\ x \leq y}} \frac{\phi_{\gamma[\mu \pm i\epsilon]}(x) \Phi_{\gamma[\mu \pm i\epsilon]}(y)}{W(\gamma[\mu \pm i\epsilon])} (1 - \frac{1}{2} \delta_{x,y}) \\ &\quad \cdot (f(x) \overline{g(y)} + f(y) \overline{g(x)}) \frac{d_q x}{p(x)} \frac{d_q y}{p(y)}, \end{aligned} \quad (7.4)$$

where $\delta_{x,y}$ is the Kronecker-delta. Let $-(1+a)^2 < \mu_1 \leq \mu \leq \mu_2 < -(1-a)^2$ and let $0 < \theta_2 \leq \theta \leq \theta_1 < \pi$ such that $\mu = \mu(e^{i\theta})$ and $\mu_j = \mu(\theta_j)$ for $j = 1, 2$. Then we have

$$\lim_{\epsilon \downarrow 0} \gamma[\mu \pm i\epsilon] = e^{\mp i\theta}. \quad (7.5)$$

Using the connection coefficients formula given in Theorem 5.9(iv) and using the fact that $W(\gamma)^{-1}$ and $\Phi_{\gamma \pm 1}(x)$ ($x \in I$) are regular at $\gamma \in \mathbb{T} \setminus \{\pm 1\}$, we obtain

$$\begin{aligned} \lim_{\epsilon \downarrow 0} \left(\frac{\phi_{\gamma[\mu+i\epsilon]}(x) \Phi_{\gamma[\mu+i\epsilon]}(y)}{W(\gamma[\mu+i\epsilon])} - \frac{\phi_{\gamma[\mu-i\epsilon]}(x) \Phi_{\gamma[\mu-i\epsilon]}(y)}{W(\gamma[\mu-i\epsilon])} \right) \\ = \frac{\phi_{e^{i\theta}}(x) \Phi_{e^{-i\theta}}(y)}{W(e^{-i\theta})} - \frac{\phi_{e^{i\theta}}(x) \Phi_{e^{i\theta}}(y)}{W(e^{i\theta})} \\ = \frac{1}{aK} \frac{\phi_{e^{i\theta}}(x) \phi_{e^{i\theta}}(y)}{c(e^{i\theta}) c(e^{-i\theta}) (e^{-i\theta} - e^{i\theta})}. \end{aligned}$$

It follows now by symmetrization of the double q -Jackson integral that

$$\begin{aligned} \lim_{\epsilon \downarrow 0} \left(\langle (L - (\mu + i\epsilon))^{-1} f, g \rangle - \langle (L - (\mu - i\epsilon))^{-1} f, g \rangle \right) \\ = \frac{1}{aK} \iint_{\substack{(x,y) \in I \times I \\ x \leq y}} \frac{\phi_{e^{i\theta}}(x) \phi_{e^{i\theta}}(y) (f(x) \overline{g(y)} + f(y) \overline{g(x)}) (1 - \frac{1}{2} \delta_{x,y})}{|c(e^{i\theta})|^2 (e^{-i\theta} - e^{i\theta})} \frac{d_q x}{p(x)} \frac{d_q y}{p(y)} \\ = \frac{1}{aK} \frac{(\mathcal{F}f)(e^{i\theta}) \overline{(\mathcal{F}g)(e^{i\theta})}}{|c(e^{i\theta})|^2 (e^{-i\theta} - e^{i\theta})}. \end{aligned}$$

The proposition follows now for all $-(1+a)^2 < \mu_1 < \mu_2 < -(1-a)^2$ using (6.6) and changing the integration variable to θ using the map $\theta \mapsto \mu(e^{i\theta})$, see (3.5). The result now also holds when $\mu_1 = -(1+a)^2$ or $\mu_2 = -(1-a)^2$ since the spectral measure E is countably additive. \square

Corollary 7.3. *The big q -Jacobi function transform \mathcal{F} uniquely extends to a continuous linear mapping $\mathcal{F}_c : \mathcal{H} \rightarrow L_W^2(\mathbb{T}, d\nu)$.*

If $(a, b, c) \in V_z^{gen}$, then $\mathcal{F}_c : \mathcal{H} \rightarrow L_W^2(\mathbb{T}, d\nu)$ factorizes through the orthogonal projection $P_c = E([- (1+a)^2, - (1-a)^2])$ (i.e. $\mathcal{F}_c = \mathcal{F}_c \circ P_c$), and the restriction of \mathcal{F}_c to the range $\mathcal{R}(P_c)$ of P_c is an isometric isomorphism onto the range $\mathcal{R}(\mathcal{F}_c) \subset L_W^2(\mathbb{T}, d\nu)$ of \mathcal{F}_c .

Proof. We assume first that $(a, b, c) \in V_z^{gen}$. In view of Proposition 7.3 applied to the special case $\mu_1 = \mu(-1) = -(1+a)^2$ and $\mu_2 = \mu(1) = -(1-a)^2$, it suffices

to observe that $E(\{\mu(\pm 1)\}) = 0$, which is a consequence of Lemma 7.1 and [20, Theorem 13.27].

Finally, observe that the inequality $\|\mathcal{F}f\|_2^2 \leq \|f\|^2$ for $f \in \mathcal{D}_{fin}$ and $(a, b, c) \in V_z^{gen}$, where $\|\cdot\|_2$ is the norm of $L_W^2(\mathbb{T}, d\nu)$, holds for all $(a, b, c) \in V$ by continuity. Hence \mathcal{F} can be uniquely extended to a continuous linear map $\mathcal{F}_c : \mathcal{H} \rightarrow L_W^2(\mathbb{C}^*, d\nu)$ for all parameters $(a, b, c) \in V$. \square

Definition 7.4. *We call \mathcal{F}_c the continuous part of the big q -Jacobi function transform.*

Observe that the limit $\langle f, \phi_\gamma \rangle_l = \lim_{k, m \rightarrow \infty} \langle f, \phi_\gamma \rangle_{k; l, m}$ exists for all $\gamma \in \mathbb{T}$ since ϕ_γ is continuously differentiable at the origin, see Lemma 3.5. It follows that

$$(\mathcal{F}_c f)(\gamma) = \lim_{l \rightarrow -\infty} \langle f, \phi_\gamma \rangle_l, \quad \gamma \in \mathbb{T} \quad \text{a.e.} \quad (7.6)$$

for $f \in \mathcal{H}$, where $\langle \cdot, \cdot \rangle_l$ ($l \in \mathbb{Z}$) is the truncated inner product

$$\langle f, g \rangle_l = \int_{-1}^{zq^l} f(x) \overline{g(x)} \frac{d_q x}{p(x)}. \quad (7.7)$$

In the remainder of this section we show that \mathcal{F}_c is surjective and we give an explicit formula for the inverse of the isometric isomorphism $\mathcal{F}_c : \mathcal{R}(P_c) \rightarrow L_W^2(\mathbb{T}, d\nu)$. The methods we employ are similar to the ones employed by Götze [8] and by Braaksma and Meulenbeld [4] for the classical Jacobi function transform, and by Kakehi [11] and Kakehi, Masuda and Ueno [12] for the little q -Jacobi function transform.

Lemma 7.5. *Let $\gamma, \delta \in \mathbb{T}$ with $\mu(\gamma) \neq \mu(\delta)$. For $k \in \mathbb{Z}_+$, $l, m \in \mathbb{Z}$ with $l < m$, we have*

$$\langle \phi_\gamma, \phi_\delta \rangle_{k; l, m} = \frac{W(\phi_\gamma, \phi_\delta)(zq^{l-1}) - W(\phi_\gamma, \phi_\delta)(zq^m) + W(\phi_\gamma, \phi_\delta)(-q^k)}{\mu(\gamma) - \mu(\delta)}.$$

Proof. For $\gamma, \delta \in \mathbb{T}$ we have that $(L\phi_\gamma)\overline{\phi_\delta} - \phi_\gamma\overline{(L\phi_\delta)} = (\mu(\gamma) - \mu(\delta))\phi_\gamma\overline{\phi_\delta}$ on I . The proof follows now from Lemma 2.2 since $\phi_\gamma : I \rightarrow \mathbb{C}$ is real valued for $\gamma \in \mathbb{T}$. \square

We define a linear map $\mathcal{G}_c : L_W^2(\mathbb{T}, d\nu) \rightarrow \mathcal{F}(I)$ by

$$(\mathcal{G}_c g)(x) = \int_{\mathbb{T}} g(\gamma) \phi_\gamma(x) d\nu(\gamma), \quad x \in I. \quad (7.8)$$

Observe that

$$\langle \mathcal{G}_c g_1, \mathcal{G}_c g_2 \rangle_{k; l, m} = \int_{\mathbb{T}} \int_{\mathbb{T}} g_1(\gamma) \overline{g_2(\gamma')} \langle \phi_\gamma, \phi_{\gamma'} \rangle_{k; l, m} d\nu(\gamma) d\nu(\gamma') \quad (7.9)$$

for $k \in \mathbb{Z}_+$, $l < m$ in \mathbb{Z} and $g_1, g_2 \in L_W^2(\mathbb{T}, d\nu)$.

Lemma 7.6. *The limit $\langle \mathcal{G}_c g_1, \mathcal{G}_c g_2 \rangle_l = \lim_{k, m \rightarrow \infty} \langle \mathcal{G}_c g_1, \mathcal{G}_c g_2 \rangle_{k; l, m}$ exists for all $g_1, g_2 \in L_W^2(\mathbb{T}, d\nu)$, and*

$$\begin{aligned} \langle \mathcal{G}_c g_1, \mathcal{G}_c g_2 \rangle_l &= \int_{\mathbb{T}} \int_{\mathbb{T}} g_1(\gamma) \overline{g_2(\gamma')} \langle \phi_\gamma, \phi_{\gamma'} \rangle_l d\nu(\gamma) d\nu(\gamma') \\ &= \int_{\mathbb{T}} \int_{\mathbb{T}} g_1(\gamma) \overline{g_2(\gamma')} \frac{W(\phi_\gamma, \phi_{\gamma'})(zq^{l-1})}{(\mu(\gamma) - \mu(\gamma'))} d\nu(\gamma) d\nu(\gamma'). \end{aligned}$$

Proof. Observe that there exists a constant $K_l > 0$ such that $|\langle \phi_\gamma, \phi_{\gamma'} \rangle_{k;l,m}| \leq K_l$ for all $\gamma, \gamma' \in \mathbb{T}$ and for all $k, m \in \mathbb{Z}_+$ with $m > l$. The first equality follows then from Lebesgue's dominated convergence theorem.

The second equality follows from Lemma 7.5, using that $W(\phi_\gamma, \phi_{\gamma'})(0^+) = W(\phi_\gamma, \phi_{\gamma'})(0^-)$ since ϕ_γ is continuously differentiable at the origin, see Lemma 3.5. \square

Finally, we determine the limit $\langle \mathcal{G}_c g_1, \mathcal{G}_c g_2 \rangle = \lim_{l \rightarrow -\infty} \langle \mathcal{G}_c g_1, \mathcal{G}_c g_2 \rangle_l$. The result is as follows.

Proposition 7.7. *The limit $\langle \mathcal{G}_c g_1, \mathcal{G}_c g_2 \rangle = \lim_{l \rightarrow -\infty} \langle \mathcal{G}_c g_1, \mathcal{G}_c g_2 \rangle_l$ exists for all $g_1, g_2 \in L_W^2(\mathbb{T}, d\nu)$, and*

$$\langle \mathcal{G}_c g_1, \mathcal{G}_c g_2 \rangle = \int_{\mathbb{T}} g_1(\gamma) \overline{g_2(\gamma)} d\nu(\gamma). \quad (7.10)$$

In particular, $\mathcal{G}_c : L_W^2(\mathbb{T}, d\nu) \rightarrow \mathcal{H}$ is an isometric isomorphism onto the range $\mathcal{R}(\mathcal{G}_c) \subset \mathcal{H}$ of \mathcal{G}_c .

Proof. We only sketch the proof, since it is similar to the little q -Jacobi case, see [12, Proposition 6.1] and [11, Proposition 7.4]. Let $C_W(\mathbb{T})$ be the algebra of complex valued, continuous, W -invariant functions on \mathbb{T} . We fix $g_1, g_2 \in C_W(\mathbb{T})$ such that $\{-1, 1\}$ is not in the support of g_1 and g_2 . It suffices to give a proof of (7.10) for such functions g_1 and g_2 . We start with the second expression of $\langle \mathcal{G}_l g_1, \mathcal{G}_l g_2 \rangle_l$ in Lemma 7.6, and replace ϕ_γ by its c -function expansion, see Theorem 5.9(iv). Using the estimate

$$\sup_{\delta \leq \theta \neq \theta' \leq \pi - \delta} \left| \frac{R_k(e^{\pm i\theta}) - R_k(e^{\pm i\theta'})}{e^{i\theta} - e^{i\theta'}} \right| = \mathcal{O}(k(qa)^k), \quad k \rightarrow \infty$$

for $0 < \delta < \pi/2$, where $R_k(\gamma) = \Phi_\gamma(zq^{-k}) - (a\gamma)^k$ (cf. (4.4)), which can easily be proved using the mean value theorem, we may replace in the expression of $\lim_{l \rightarrow -\infty} \langle \mathcal{G}_c g_1, \mathcal{G}_c g_2 \rangle_l$ the function $\Phi_\gamma(zq^{l-1})$ by its asymptotic value $(a\gamma)^{1-l}$ at ∞ . Combined with (2.15) it follows now from the bounded convergence theorem that

$$\lim_{m \rightarrow \infty} \langle \mathcal{G}_c g_1, \mathcal{G}_c g_2 \rangle_{1-m} = \frac{-a}{4\pi^2 K} \lim_{m \rightarrow \infty} \int_0^\pi \int_0^\pi \frac{g_1(e^{i\theta}) \overline{g_2(e^{i\theta'})} s_m(\theta, \theta') d\theta d\theta'}{|c(e^{i\theta})|^2 |c(e^{i\theta'})|^2 (\mu(e^{i\theta'}) - \mu(e^{i\theta}))}$$

provided that the limit in the right hand side of the equality exists, with

$$s_m(\theta, \theta') = \sum_{\epsilon, \xi \in \{\pm 1\}} c(e^{i\epsilon\theta}) c(e^{i\xi\theta'}) e^{i(m-1)(\epsilon\theta + \xi\theta')} (e^{i\epsilon\theta} - e^{i\xi\theta'}).$$

Since $c(\gamma)$ is continuous and non-zero on $\mathbb{T} \setminus \{\pm 1\}$, the Riemann-Lebesgue lemma implies that the contributions of the sums of s_m with $\epsilon\xi > 0$ tend to zero in the limit, so $s_m(\theta, \theta')$ may be replaced by

$$\begin{aligned} t_m(\theta, \theta') &= c(e^{i\theta}) c(e^{-i\theta'}) e^{i(m-1)(\theta - \theta')} (e^{i\theta} - e^{-i\theta'}) \\ &\quad + c(e^{-i\theta}) c(e^{i\theta'}) e^{i(m-1)(\theta' - \theta)} (e^{-i\theta} - e^{i\theta'}) \\ &= -4c(e^{i\theta}) c(e^{-i\theta'}) \sin\left(\frac{\theta + \theta'}{2}\right) \sin\left(\frac{(2m-1)(\theta - \theta')}{2}\right) \\ &\quad + (c(e^{-i\theta}) c(e^{i\theta'}) - c(e^{i\theta}) c(e^{-i\theta'})) e^{i(m-1)(\theta' - \theta)} (e^{-i\theta} - e^{i\theta'}). \end{aligned}$$

Applying the Riemann-Lebesgue lemma again and using

$$\mu(e^{i\theta'}) - \mu(e^{i\theta}) = 2a(\cos(\theta') - \cos(\theta)) = 4a \sin\left(\frac{\theta + \theta'}{2}\right) \sin\left(\frac{\theta - \theta'}{2}\right),$$

we arrive at

$$\lim_{m \rightarrow \infty} \langle \mathcal{G}_c g_1, \mathcal{G}_c g_2 \rangle_{1-m} = \lim_{m \rightarrow \infty} \frac{1}{4\pi^2 K} \int_0^\pi \int_0^\pi \frac{g_1(e^{i\theta}) \overline{g_2(e^{i\theta'})}}{c(e^{-i\theta}) c(e^{i\theta'})} D_m(\theta, \theta') d\theta d\theta'$$

provided that the limit in the right hand side of the equality exists, where D_m is the Dirichlet kernel,

$$D_m(\theta, \theta') = \frac{\sin((2m-1)(\theta - \theta')/2)}{\sin((\theta - \theta')/2)}.$$

The result follows now from the well known L^2 -properties of the Dirichlet kernel. \square

Recall the notation P_c for the orthogonal projection $E([- (1+a)^2, -(1-a)^2])$.

Proposition 7.8. (i) $\mathcal{F}_c|_{\mathcal{R}(\mathcal{G}_c)} : \mathcal{R}(\mathcal{G}_c) \rightarrow L_W^2(\mathbb{T}, d\nu)$ is a surjective isometric isomorphism. Its inverse is given by $\mathcal{G}_c : L_W^2(\mathbb{T}, d\nu) \rightarrow \mathcal{R}(\mathcal{G}_c)$.

(ii) We have $\mathcal{R}(\mathcal{G}_c) = \mathcal{R}(P_c)$ for $(a, b, c) \in V_z^{gen}$.

Proof. (i) It follows from Corollary 7.3, Lemma 7.6 and Proposition 7.7 that

$$\begin{aligned} \int_{\mathbb{T}} (\mathcal{F}_c(\mathcal{G}_c f))(\gamma) \overline{g(\gamma)} d\nu(\gamma) &= \lim_{l \rightarrow -\infty} \lim_{k, l \rightarrow \infty} \int_{\mathbb{T}} \langle \mathcal{G}_c f, \phi_\gamma \rangle_{k;l,m} \overline{g(\gamma)} d\nu(\gamma) \\ &= \lim_{l \rightarrow -\infty} \lim_{k, l \rightarrow \infty} \langle \mathcal{G}_c f, \mathcal{G}_c g \rangle_{k;l,m} = \int_{\mathbb{T}} f(\gamma) \overline{g(\gamma)} d\nu(\gamma) \end{aligned}$$

for all $f, g \in L_W^2(\mathbb{T}, d\nu)$. Hence $\mathcal{F}_c \circ \mathcal{G}_c$ is the identity on $L_W^2(\mathbb{T}, d\nu)$ and $\mathcal{R}(\mathcal{F}_c) = L_W^2(\mathbb{T}, d\nu)$. Consequently, we have $\mathcal{G}_c \circ \mathcal{F}_c|_{\mathcal{R}(\mathcal{G}_c)} = \text{Id}_{\mathcal{R}(\mathcal{G}_c)}$.

(ii) Assume that $(a, b, c) \in V_z^{gen}$. Let $f \in \mathcal{R}(\mathcal{G}_c)$. Observe that $\|f\| = \|\mathcal{F}_c f\|_2$ by the previous paragraph and by Proposition 7.7, where $\|\cdot\|_2$ is the norm of $L_W^2(\mathbb{T}, d\nu)$. By Corollary 7.3, this implies that $f \in \mathcal{R}(P_c)$.

Let $f \in \mathcal{R}(P_c)$. We have seen that $\mathcal{F}_c(\mathcal{R}(\mathcal{G}_c)) = L_W^2(\mathbb{T}, d\nu)$, hence there exists a function $g \in \mathcal{R}(\mathcal{G}_c)$ such that $\mathcal{F}_c f = \mathcal{F}_c g$. Now $f - g \in \mathcal{R}(P_c)$ by the previous paragraph, and $\mathcal{F}_c|_{\mathcal{R}(P_c)}$ is injective by Corollary 7.3, hence $f = g \in \mathcal{R}(\mathcal{G}_c)$, as desired. \square

8. THE POINT SPECTRUM

In this section we determine the resolution of the identity E on $\mathbb{R} \setminus [-(1+a)^2, -(1-a)^2]$. Observe that the kernel $K_\gamma(x, y)$ of the Green function is meromorphic as function of $\gamma \in \mathbb{D}$, where $\mathbb{D} = \{v \in \mathbb{C} \mid 0 < |v| < 1\}$ is the punctured open unit disc in the complex plane. Since $\phi_\gamma(x)$ and $\Phi_\gamma(x)$ are analytic at $\gamma \in \mathbb{D}$, see Theorem 5.9(ii) for Φ_γ , we have that the poles of $K_\gamma(x, y)$ coincide with the poles of $W(\gamma)^{-1}$, which in turn coincide with the zeros of the map $\gamma \mapsto c(\gamma^{-1})$ ($\gamma \in \mathbb{D}$), where $c(\cdot)$ is the c -function defined by (4.7). In particular, the poles of $K_\gamma(x, y)$ in

$\gamma \in \mathbb{D}$ are independent of $x, y \in I$ and are given by the set

$$S = \left\{ \frac{1}{q^k e} \mid e \in \{a, b, c\}, k \in \mathbb{Z}_+ : \frac{1}{q^k e} < 1 \right\} \cup \left\{ -\frac{abc z}{q^{k+1}} \mid k \in \mathbb{Z} : \frac{abc z}{q^{k+1}} < 1 \right\}. \quad (8.1)$$

Observe that these poles of $K_\gamma(x, y)$ are simple since at most one of the parameters a, b, c is ≥ 1 . Furthermore, observe that $S \subset S_{reg}$ when $(a, b, c) \in V_z^{gen}$.

Lemma 8.1. *The function $\phi_{\tilde{\gamma}}$ ($\tilde{\gamma} \in S$) is an eigenfunction of (L, \mathcal{D}) with eigenvalue $\mu(\tilde{\gamma})$. In particular, $\mu(S)$ is contained in the point spectrum $\sigma_p(L)$ of (L, \mathcal{D}) .*

Proof. Let $\tilde{\gamma} \in S$. Then $\phi_{\tilde{\gamma}} \in V_{\mu(\tilde{\gamma})}$ by Lemma 3.5. Furthermore, if $\tilde{\gamma} \notin S_{reg}$, then

$$\phi_{\tilde{\gamma}}|_{I_+} = c(\tilde{\gamma})\Phi_{\tilde{\gamma}} \in M_{\mu(\tilde{\gamma})}, \quad (8.2)$$

where the first equality follows from Proposition 4.4. Since $c(\gamma)$ and $\Phi_\gamma \in V_{\mu(\gamma)}^+$ are regular at $\gamma \in S \cap S_{reg}$, it follows that (8.2) is valid for all $\tilde{\gamma} \in S$. Hence $\phi_{\tilde{\gamma}} \in \mathcal{D}$ and $L\phi_{\tilde{\gamma}} = \mu(\tilde{\gamma})\phi_{\tilde{\gamma}}$ on I for all $\tilde{\gamma} \in S$, as desired. \square

Remark 8.2. If $(a, b, c) \in V_z^{gen}$, then $S \subset S_{reg}$ and $\phi_{\tilde{\gamma}}|_{I_+} = c(\tilde{\gamma})\Phi_{\tilde{\gamma}}$ by Theorem 5.9(iv), so formula (8.2) extends for these parameter values to the whole q -interval I .

Let $\tilde{\gamma} \in S$. It follows from Lemma 8.1 that the linear functional $f \mapsto (\mathcal{F}f)(\tilde{\gamma})$ with $f \in \mathcal{D}_{fin}$ is bounded, where \mathcal{F} is the big q -Jacobi function transform defined by (7.1), hence it uniquely extends to a continuous linear functional on \mathcal{H} . It is given explicitly by

$$(\mathcal{F}f)(\tilde{\gamma}) = \langle f, \phi_{\tilde{\gamma}} \rangle, \quad f \in \mathcal{H}. \quad (8.3)$$

We define the weight $d\nu(\{\tilde{\gamma}\})$ for $\tilde{\gamma} \in S$ and $(a, b, c) \in V_z^{gen}$ by

$$d\nu(\{\tilde{\gamma}\}) = \frac{1}{K} \text{Res}_{\gamma=\tilde{\gamma}} \left(\frac{-1}{\gamma c(\gamma) c(\gamma^{-1})} \right) = \frac{1}{K} \text{Res}_{\gamma=\tilde{\gamma}^{-1}} \left(\frac{1}{\gamma c(\gamma) c(\gamma^{-1})} \right), \quad (8.4)$$

where K is the positive constant defined by (2.14). Observe that the poles $\tilde{\gamma} \in S$ of the meromorphic function $\gamma \mapsto (c(\gamma)c(\gamma^{-1}))^{-1}$ are simple for parameters $(a, b, c) \in V_z^{gen}$. We derive explicit expressions for the discrete weights $d\nu(\{\tilde{\gamma}\})$ by relating the continuous part of the Plancherel measure with the weight function $\Delta(x) = \Delta(x; t_0, t_1, t_2, t_3)$ of the Askey-Wilson polynomials,

$$\Delta(x) = \frac{(x^2, 1/x^2; q)_\infty}{\prod_{j=0}^3 (t_j x, t_j/x; q)_\infty}. \quad (8.5)$$

Observe that

$$\frac{1}{K c(\gamma) c(\gamma^{-1}) \gamma} = M \frac{\Delta(\gamma; e, f, -q/efgz, -efgz)}{(g\gamma, g/\gamma; q)_\infty \gamma}, \quad (8.6)$$

where $\{e, f, g\}$ is an arbitrary permutation of $\{a, b, c\}$ (taking multiplicity into account), and where $M = M(a, b, c; z)$ is the positive constant

$$M = \frac{1}{K} (ab, ac; q)_\infty^2 \theta(-bcz, -bcz) = \frac{(ab, ac; q)_\infty^2 \theta(-abz, -acz, -bcz)}{(1-q)z\theta(-1/z)}. \quad (8.7)$$

For a simple pole eq^k of $\Delta(x; t_0, t_1, t_2, t_3)$, where $e \in \{t_j\}_{j=0}^3$ and $k \in \mathbb{Z}_+$, we have the explicit formula

$$\begin{aligned} \operatorname{Res}_{\gamma=eq^k} \left(\frac{\Delta(x; t_0, t_1, t_2, t_3)}{x} \right) &= \frac{(e^{-2}; q)_\infty}{(q, ef, f/e, eg, g/e, eh, h/e; q)_\infty} \\ &\cdot \frac{(e^2, ef, eg, eh; q)_k}{(q, qe/f, qe/g, qe/h; q)_k} \frac{(1 - e^2 q^{2k})}{(1 - e^2)} \left(\frac{q}{efgh} \right)^k, \end{aligned} \quad (8.8)$$

see [7, (7.5.22)], where $\{f, g, h\}$ is such that $\{e, f, g, h\} = \{t_0, t_1, t_2, t_3\}$ (taking multiplicity into account). It is well known that the right hand side of (8.8) is well defined and positive for real parameters t_i such that the $t_i t_j$ ($i \neq j$) are strictly less than one.

Combined with (8.4), (8.6) and (2.12), we obtain for $e \in \{a, b, c\}$ and $k \in \mathbb{Z}_+$ such that $1/q^k e < 1$,

$$\begin{aligned} d\nu(\{1/q^k e\}) &= M \frac{(e^{-2}; q)_\infty}{(q, ef, f/e, eg, g/e; q)_\infty} \frac{\theta(-fgz, -e^2 fgz)}{\theta(-fgz, -e^2 fgz)} \\ &\cdot \frac{(e^2, ef, eg; q)_k}{(q, qe/f, qe/g; q)_k} \frac{(1 - e^2 q^{2k})}{(1 - e^2)} \left(\frac{-q^{(k+1)/2}}{fg} \right)^k, \end{aligned} \quad (8.9)$$

where f, g are such that $\{e, f, g\} = \{a, b, c\}$ (taking multiplicity into account). By the positivity of the residues of the Askey-Wilson measure, it follows that the right hand side of (8.9) is well defined, regular and strictly positive for parameters $(a, b, c) \in V$.

Similarly, we obtain for $k \in \mathbb{Z}$ such that $abc z/q^{k+1} < 1$,

$$\begin{aligned} d\nu(\{-abc z/q^{k+1}\}) &= \\ &= \frac{M}{(q, q, -q/abz, -q/acz, -q/bcz, -a^2bcz/q, -ab^2cz/q, -abc^2z/q; q)_\infty} \\ &\cdot \frac{(-q/abz, -q/acz, -q/bcz; q)_k}{(-1/a^2bc, -1/ab^2c, -1/abc^2; q)_k} \left(q^{2k} - \frac{a^2 b^2 c^2 z^2}{q^2} \right) \left(\frac{-q^{(k+3)/2}}{a^2 b^2 c^2 z} \right)^k. \end{aligned} \quad (8.10)$$

The right hand side of (8.10) is well defined, regular and strictly positive for parameters $(a, b, c) \in V$. We will sometimes abuse notation by writing $\nu(\{\tilde{\gamma}\})$ for the discrete weight $d\nu(\{\gamma\})$ ($\tilde{\gamma} \in S$).

Proposition 8.3. *Let $(a, b, c) \in V_z^{gen}$.*

(i) *For $\mu_1 < \mu_2 < -(1+a)^2$ or $-(1-a)^2 < \mu_1 < \mu_2$ such that $\mu(S) \cap (\mu_1, \mu_2) = \emptyset$, we have $E((\mu_1, \mu_2)) = 0$.*

(ii) *For $\tilde{\gamma} \in S$ and $f, g \in \mathcal{H}$ we have*

$$\langle E(\{\mu(\tilde{\gamma})\}) f, g \rangle = (\mathcal{F}f)(\tilde{\gamma}) \overline{(\mathcal{F}g)(\tilde{\gamma})} d\nu(\{\tilde{\gamma}\}).$$

Proof. Throughout the proof, we use the notations introduced in the proof of Proposition 7.2.

(i) Let $f, g \in \mathcal{D}_{fin}$ and fix $\mu_1 < \mu_2$ satisfying the properties as stated in (i). It suffices to prove that $\langle E((\mu_1, \mu_2)) f, g \rangle = 0$. Observe that (7.4) is still valid in the present setting, but that the analogue of (7.5) is now given by

$$\lim_{\epsilon \downarrow 0} \gamma[\mu \pm i\epsilon] = \gamma, \quad (8.11)$$

where $\mu_1 < \mu < \mu_2$ and where $\gamma \in \mathbb{D}$ is the unique element satisfying $\mu = \mu(\gamma)$. By the condition $\mu(S) \cap (\mu_1, \mu_2) = \emptyset$, we have for $\gamma \in \mathbb{D}$ satisfying $\mu_1 < \mu(\gamma) < \mu_2$ that

$$\lim_{\epsilon \downarrow 0} \langle (L - (\mu(\gamma) \pm i\epsilon))^{-1} f, g \rangle = \iint_{\substack{(x,y) \in I \times I \\ x \leq y}} \frac{\phi_\gamma(x) \Phi_\gamma(y)}{W(\gamma)} \left(1 - \frac{1}{2} \delta_{x,y}\right) \cdot (f(x) \overline{g(y)} + f(y) \overline{g(x)}) \frac{d_q x}{p(x)} \frac{d_q y}{p(y)}. \quad (8.12)$$

By the bounded convergence theorem we may interchange the limit $\epsilon \downarrow 0$ and the integration in (6.6). Combined with (8.12), this gives $\langle E((\mu_1, \mu_2)) f, g \rangle = 0$, as desired.

(ii) Let $\tilde{\gamma} \in S$ and $f, g \in \mathcal{D}_{fin}$. Choose arbitrary $\mu_1 < \mu_2$ such that $\mu(S) \cap (\mu_1, \mu_2) = \mu(\tilde{\gamma})$ and such that $[\mu_1, \mu_2] \cap [-(1+a)^2, -(1-a)^2] = \emptyset$. Then by the first part of the proposition, we have $E(\{\mu(\tilde{\gamma})\}) = E((\mu_1, \mu_2))$. We compute now $\langle E((\mu_1, \mu_2)) f, g \rangle$ using (6.6). We substitute (7.4) in (6.6) and change the integration parameter μ in (6.6) to the γ -parameter using (3.5). Observe that for $\mu_1 < \mu < \mu_2$ and $\epsilon > 0$, we have that $\gamma[\mu + i\epsilon]$ (respectively $\gamma[\mu - i\epsilon]$) lies in the lower (respectively upper) half plane of \mathbb{C} . Since $W(\gamma)^{-1}$ has a simple pole in $\tilde{\gamma}$, it follows by Cauchy's Theorem that

$$\begin{aligned} \langle E((\mu_1, \mu_2)) f, g \rangle &= \iint_{\substack{(x,y) \in I \times I \\ x \leq y}} (f(x) \overline{g(y)} + f(y) \overline{g(x)}) \left(1 - \frac{1}{2} \delta_{x,y}\right) \\ &\quad \cdot \phi_{\tilde{\gamma}}(x) \Phi_{\tilde{\gamma}}(y) \left(\frac{a}{\tilde{\gamma}} (\tilde{\gamma}^{-1} - \tilde{\gamma}) \operatorname{Res}_{\gamma=\tilde{\gamma}}(W(\gamma)^{-1}) \right) \frac{d_q x}{p(x)} \frac{d_q y}{p(y)}, \end{aligned}$$

where the factor $\frac{a}{\tilde{\gamma}} (\tilde{\gamma}^{-1} - \tilde{\gamma})$ arises from changing the integration variable in (6.6) to γ using the map $\gamma \mapsto \mu(\gamma)$, and from the fact that one has to change sign in order to get a positive oriented curve around $\tilde{\gamma}$. The proof is now completed using the explicit expression of $W(\gamma)$, using Remark 8.2 and by symmetrizing the double q -Jackson integral. \square

Corollary 8.4. *Let $(a, b, c) \in V_z^{gen}$. The spectrum $\sigma(L)$ of the self-adjoint operator (L, \mathcal{D}) is given by*

$$\sigma(L) = [-(1+a)^2, -(1-a)^2] \cup \mu(S),$$

where $\sigma_c(L) = [-(1+a)^2, -(1-a)^2]$ is the continuous spectrum and $\sigma_p(L) = \mu(S)$ is the point spectrum.

Proof. It follows from [20, Theorem 13.27], Proposition 7.2, Proposition 7.8, Lemma 8.1 and Proposition 8.3 that $\sigma(L) = [-(1+a)^2, -(1-a)^2] \cup \mu(S)$. Observe now that $[-(1+a)^2, -(1-a)^2] \subset \sigma_c(L)$ by Corollary 7.1 and $\mu(S) \subset \sigma_p(L)$ by Lemma 8.1. It follows that $\sigma_c(L) = [-(1+a)^2, -(1-a)^2]$ and $\sigma_p(L) = \mu(S)$, since $\sigma(L)$ is the disjoint union of $\sigma_c(L)$ and $\sigma_p(L)$, see [20, Theorem 13.27]. \square

Corollary 8.5. *The functions $\phi_{\tilde{\gamma}} \in \mathcal{D} \subset \mathcal{H}$ ($\tilde{\gamma} \in S$) are mutually orthogonal in \mathcal{H} . Their quadratic norms are given by $\|\phi_{\tilde{\gamma}}\|^2 = \nu(\{\tilde{\gamma}\})^{-1}$.*

Proof. Orthogonality is clear since the functions $\phi_{\tilde{\gamma}}$ ($\tilde{\gamma} \in S$) are eigenfunctions of the self-adjoint operator (L, \mathcal{D}) with mutually different eigenvalues $\mu(\tilde{\gamma})$ ($\tilde{\gamma} \in S$), see Lemma 8.1.

It remains to derive the explicit expression for the quadratic norm $\|\phi_{\tilde{\gamma}}\|^2$. We first assume that $(a, b, c) \in V_z^{gen}$. Observe that $E(\{\mu(\tilde{\gamma})\})\phi_{\tilde{\gamma}} = \nu(\{\tilde{\gamma}\})\|\phi_{\tilde{\gamma}}\|^2\phi_{\tilde{\gamma}}$ for $\tilde{\gamma} \in S$ by Proposition 8.3(ii). Since $\nu(\{\tilde{\gamma}\})\|\phi_{\tilde{\gamma}}\|^2 \neq 0$ and $E(\{\mu(\tilde{\gamma})\})$ is a projection, it follows that $\|\phi_{\tilde{\gamma}}\|^2 = \nu(\{\tilde{\gamma}\})^{-1}$ for $\tilde{\gamma} \in S$, as desired. The result now follows for $(a, b, c) \in V$ by continuity. \square

9. THE PLANCHEREL FORMULA, INVERSION FORMULA AND THE DUAL ORTHOGONALITY RELATIONS

The explicit knowledge of the resolution of the identity E of the self-adjoint operator (L, \mathcal{D}) on \mathcal{H} leads directly to the Plancherel formula and the inversion formula for the big q -Jacobi function transform, which we formulate in this section explicitly. We show that the dual orthogonality relations imply orthogonality relations for the continuous dual q^{-1} -Hahn polynomials with respect to a one-parameter family of non-extremal weight functions. Furthermore, the dual orthogonality relations give explicit sets of functions which complete the continuous dual q^{-1} -Hahn polynomials to orthogonal bases of the corresponding L^2 -space.

We define the measure $d\nu(\cdot) = d\nu(\cdot; a, b, c; z)$ on \mathbb{C}^* by

$$\int_{\mathbb{C}^*} f(\gamma) d\nu(\gamma) = \int_{\mathbb{T}} f(\gamma) d\nu(\gamma) + \sum_{\tilde{\gamma} \in S} f(\tilde{\gamma}) d\nu(\{\tilde{\gamma}\}), \quad (9.1)$$

where the measure $d\nu(\gamma)$ on \mathbb{T} is defined by (7.2), $S \subset \mathbb{D}$ is the discrete set defined by (8.1) and the point mass $d\nu(\{\tilde{\gamma}\})$ for $\tilde{\gamma} \in S$ is defined by (8.9) for $\tilde{\gamma} > 0$ and (8.10) for $\tilde{\gamma} < 0$.

The measure $d\nu(\cdot)$ on \mathbb{C}^* is well defined for $(a, b, c) \in V$. Indeed, for the absolutely continuous part of the measure the conditions on the parameters are such that the possible zeros of the denominator of the corresponding weight function $1/|c(\cdot)|^2$ are compensated by zeros of the numerator. For the discrete part of the measure it follows from the explicit expressions (8.9) and (8.10) that $d\nu(\{\tilde{\gamma}\})$ is well defined and strictly positive for all $\tilde{\gamma} \in S$.

Let $L_W^2(\mathbb{C}^*, d\nu)$ be the Hilbert space of W -invariant L^2 -functions with respect to the measure $d\nu$. We define the big q -Jacobi function transform for functions $f \in \mathcal{H}$ such that $f(zq^{-k}) = 0$ for $k \gg 0$ by

$$g(\gamma) := (\mathcal{F}f)(\gamma) = \langle f, \phi_{\gamma} \rangle = \int_{-1}^{\infty(z)} f(x) \overline{\phi_{\gamma}(x)} \frac{d_q x}{p(x)}, \quad \gamma \in \mathbb{C}^*, \quad (9.2)$$

and we define for functions $g \in L_W^2(\mathbb{C}^*, d\nu)$ satisfying $g(-q^k abc z) = 0$ for $k \gg 0$,

$$f(x) := (\mathcal{G}g)(x) = \int_{\mathbb{C}^*} g(\gamma) \phi_{\gamma}(x) d\nu(\gamma), \quad x \in I. \quad (9.3)$$

The results in section 7 and section 8 lead to the following main theorem of this paper.

Theorem 9.1 (The big q -Jacobi function transform). *Let $z > 0$ and $(a, b, c) \in V$. The maps \mathcal{F} and \mathcal{G} uniquely extend to surjective isometric isomorphisms*

$$\mathcal{F} : \mathcal{H} \rightarrow L_W^2(\mathbb{C}^*, d\nu), \quad \mathcal{G} : L_W^2(\mathbb{C}^*, d\nu) \rightarrow \mathcal{H}.$$

Furthermore, $\mathcal{G} = \mathcal{F}^{-1}$, hence (9.2) and (9.3) give the big q -Jacobi function transform pair (interpreted in the suitable L^2 -sense).

Proof. We write (\cdot, \cdot) for the inner product of $L_W^2(\mathbb{C}^*, d\nu)$.

Suppose that $(a, b, c) \in V_z^{gen}$ and let $f, g \in \mathcal{D}_{fin}$. Applying Proposition 7.2 and Proposition 8.3, we obtain $\langle f, g \rangle = \langle E(\mathbb{R})f, g \rangle = (\mathcal{F}f, \mathcal{F}g)$. In order to extend this result to parameters (a, b, c) in V , we show that $\langle f, g \rangle$ and $(\mathcal{F}f, \mathcal{F}g)$ depend continuously on $(a, b, c) \in V$. This is clear for $\langle f, g \rangle$, while for $(\mathcal{F}f, \mathcal{F}g)$ it is clear except for the term

$$\sum_{\tilde{\gamma} \in S: \tilde{\gamma} < 0} (\mathcal{F}f)(\tilde{\gamma}) \overline{(\mathcal{F}g)(\tilde{\gamma})} d\nu(\tilde{\gamma}) = \sum_{k \in \mathbb{Z}} (\mathcal{F}f)(-q^k abcz) \overline{(\mathcal{F}g)(-q^k abcz)} d\nu(-q^k abcz),$$

where we use the convention that $d\nu(-q^k abcz) = 0$ if $-q^k abcz \notin S$ for the right hand side. The continuity of this term follows by Lebesgue's dominated convergence theorem, using the asymptotics

$$\phi_{-q^k abcz}(x) = \mathcal{O}(x^{-k}), \quad d\nu(-q^k abcz) = \mathcal{O}(z^k q^{k(k-1)/2}), \quad k \rightarrow \infty, \quad (9.4)$$

where $x \in I$, which hold uniformly for (a, b, c) in compacta of V . To compute the asymptotics (9.4) for $\phi_\gamma(x)$ with $x \in I_+$ we used (8.2), and for $x \in I_-$ we used the formula

$$\phi_\gamma(x) = \frac{(q/ab, q/ac, q\gamma^2; q)_\infty}{(q\gamma/a, q\gamma/b, q\gamma/c; q)_\infty} \Phi_\gamma^-(x), \quad (9.5)$$

see Lemma 4.5 (observe that the right hand side of (9.5) is well defined for $\gamma < 0$ with $|\gamma| < a^{-1}$ and can be uniquely extended by continuity to $(a, b, c) \in V$).

It follows that $\langle f, g \rangle = (\mathcal{F}f, \mathcal{F}g)$ for $f, g \in \mathcal{D}_{fin}$ and $(a, b, c) \in V$, hence \mathcal{F} uniquely extends to an isometric isomorphism $\mathcal{F} : \mathcal{H} \rightarrow L_W^2(\mathbb{C}^*, d\nu)$ onto its image for all $(a, b, c) \in V$.

Let f and g be W -invariant continuous functions on \mathbb{C}^* with compact support. We have $(\mathcal{G}f)(x) = (\mathcal{G}_c f)(x) + (\mathcal{G}_d f)(x)$ for $x \in I$ with \mathcal{G}_c given by (7.8) and

$$(\mathcal{G}_d f)(x) = \sum_{\tilde{\gamma} \in S} f(\tilde{\gamma}) \phi_{\tilde{\gamma}}(x) d\nu(\{\tilde{\gamma}\}),$$

which is a finite sum by the assumptions on f . Assume that $(a, b, c) \in V_z^{gen}$, then it follows from Proposition 7.8(ii) that $\mathcal{G}_c f \in \mathcal{R}(P_c)$. Furthermore, it follows from the proof of Corollary 8.5 that $\mathcal{G}_d f \in \mathcal{R}(P_d) = (\mathcal{R}(P_c))^\perp$, where $P_d = E(\mu(S))$. Hence $\mathcal{G}f \in \mathcal{H}$ and $\langle \mathcal{G}f, \mathcal{G}g \rangle = \langle \mathcal{G}_c f, \mathcal{G}_c g \rangle + \langle \mathcal{G}_d f, \mathcal{G}_d g \rangle$. It follows from Proposition 7.8 and Corollary 8.5 that

$$\langle \mathcal{G}_c f, \mathcal{G}_c g \rangle = \int_{\gamma \in \mathbb{T}} f(\gamma) \overline{g(\gamma)} d\nu(\gamma), \quad \langle \mathcal{G}_d f, \mathcal{G}_d g \rangle = \sum_{\tilde{\gamma} \in S} f(\tilde{\gamma}) \overline{g(\tilde{\gamma})} d\nu(\{\tilde{\gamma}\}). \quad (9.6)$$

This implies that $\langle \mathcal{G}f, \mathcal{G}g \rangle = (f, g)$.

It follows from Proposition 7.8(i) and Corollary 8.5 that (9.6) is valid for all $(a, b, c) \in V$. Furthermore, by continuity arguments, we have $\langle \mathcal{G}_c f, \mathcal{G}_d g \rangle = 0$ for all $(a, b, c) \in V$. Hence, \mathcal{G} uniquely extends to an isometric isomorphism $\mathcal{G} : L_W^2(\mathbb{C}^*, d\nu) \rightarrow \mathcal{H}$ onto its image for all $(a, b, c) \in V$.

A direct computation now shows that

$$(\mathcal{F}f, g) = \langle f, \mathcal{G}g \rangle, \quad \forall f \in \mathcal{H}, \quad \forall g \in L_W^2(\mathbb{C}^*, d\nu)$$

for $(a, b, c) \in V$, cf. the proof of Proposition 7.8. This implies that $\mathcal{R}(\mathcal{F}) = L_W^2(\mathbb{C}^*, d\nu)$, $\mathcal{R}(\mathcal{G}) = \mathcal{H}$ and $\mathcal{G} = \mathcal{F}^{-1}$, which completes the proof of the theorem. \square

In the remainder of this section we derive a one-parameter family of non-extremal orthogonality measures for the continuous dual q^{-1} -Hahn polynomials, and explicit sets of functions which complement the polynomials to orthogonal bases of the corresponding Hilbert spaces.

We write $t_0 = 1/a$, $t_1 = 1/b$, $t_2 = 1/c$. The condition $(a, b, c) \in V$ is then equivalent to the conditions

$$t_i > 0, \quad t_i t_j > 1 \quad (i \neq j) \quad (9.7)$$

on the parameters (t_0, t_1, t_2) . We define polynomials $p_k(\gamma) = p_k(\gamma; t_0, t_1, t_2; q^{-1})$ ($k \in \mathbb{Z}_+$) in $\gamma + \gamma^{-1}$ by

$$p_k(\gamma) = \phi_\gamma(-q^k; t_0^{-1}, t_1^{-1}, t_2^{-1}) = {}_3\phi_2 \left(\begin{matrix} q^k, t_0\gamma, t_0/\gamma \\ t_0t_1, t_0t_2 \end{matrix}; q^{-1}, q^{-1} \right), \quad k \in \mathbb{Z}_+. \quad (9.8)$$

Here we use [7, Exercise 1.4(i)] to obtain the second equality. Observe that the second equality of (9.8) shows that $\{p_k(\cdot; t_0, t_1, t_2; q^{-1})\}_{k \in \mathbb{Z}_+}$ are exactly the continuous dual q^{-1} -Hahn polynomials, i.e. Askey-Wilson polynomials in base q^{-1} with one of the four parameters equal to zero. For $z > 0$, we define a measure $d\sigma_z(\cdot) = d\sigma_z(\cdot; t_0, t_1, t_2; q^{-1})$ on \mathbb{C}^* by

$$\int_{\mathbb{C}^*} f(\gamma) d\sigma_z(\gamma) = \frac{1}{M} \int_{\mathbb{C}^*} f(\gamma) d\nu(\gamma; t_0^{-1}, t_1^{-1}, t_2^{-1}; z). \quad (9.9)$$

Explicitly, we have

$$\int_{\mathbb{C}^*} f(\gamma) d\sigma_z(\gamma) = \frac{1}{4\pi i} \int_{\mathbb{T}} f(\gamma) w_z(\gamma) \frac{d\gamma}{\gamma} + \sum_{\tilde{\gamma} \in S_z} f(\tilde{\gamma}) \operatorname{Res}_{\gamma=\tilde{\gamma}} \left(\frac{w_z(\gamma)}{\gamma} \right), \quad (9.10)$$

where

$$S_z = S^{-1} = \left\{ \frac{q^k}{e} \mid e \in \{t_0, t_1, t_2\}, k \in \mathbb{Z}_+ : \frac{q^k}{e} > 1 \right\} \\ \cup \left\{ \frac{-q^k t_0 t_1 t_2}{z} \mid k \in \mathbb{Z} : \frac{q^k t_0 t_1 t_2}{z} > 1 \right\}$$

and with weight function $w_z(\cdot) = w_z(\cdot; t_0, t_1, t_2; q^{-1})$ given by

$$w_z(\gamma) = \frac{(\gamma^2, 1/\gamma^2; q)_\infty}{\theta(-z\gamma/t_0 t_1 t_2, -z/t_0 t_1 t_2 \gamma) \prod_{j=0}^2 (\gamma/t_j, 1/t_j \gamma; q)_\infty} \quad (9.11)$$

for parameters (t_0, t_1, t_2) such that the poles of $w_z(\gamma)$ at $\gamma \in S_z$ are simple. Finally, we define W -invariant functions $r_k^z(\gamma) = r_k^z(\gamma; t_0, t_1, t_2; q^{-1})$ for $k \in \mathbb{Z}$ by

$$r_k^z(\gamma) = \phi_\gamma(zq^k; t_0^{-1}, t_1^{-1}, t_2^{-1}) \\ = \frac{(\gamma/t_0, 1/t_1 t_2, -q^k z/t_0 t_1 t_2 \gamma; q)_\infty}{(1/t_0 t_1, 1/t_0 t_2, -q^k z/t_1 t_2; q)_\infty} {}_3\phi_2 \left(\begin{matrix} 1/t_1 \gamma, 1/t_2 \gamma, -q^k z/t_1 t_2 \\ 1/t_1 t_2, -q^k z/t_0 t_1 t_2 \gamma \end{matrix}; q, \gamma/t_0 \right) \quad (9.12)$$

where the second equality holds for $\gamma \in \mathbb{C}^*$ with $|\gamma/t_0| < 1$, see (3.8).

Theorem 9.2. *Let $z > 0$ and fix parameters t_i satisfying the conditions (9.7). Then, $\{p_k\}_{k \in \mathbb{Z}_+} \cup \{r_k^z\}_{k \in \mathbb{Z}}$ is an orthogonal basis of the Hilbert space $L_W^2(\mathbb{C}^*, d\sigma_z)$. The quadratic norms of the basis elements are given by*

$$\begin{aligned} \int_{\mathbb{C}^*} |p_k(\gamma)|^2 d\sigma_z(\gamma) &= \frac{\theta(-z)}{\theta(-z/t_0 t_1, -z/t_0 t_2, -z/t_1 t_2)} \\ &\quad \cdot \frac{1}{(q, 1/t_0 t_1, 1/t_0 t_2, 1/t_1 t_2; q)_\infty} \frac{(q, 1/t_1 t_2; q)_k}{(1/t_0 t_1, 1/t_0 t_2; q)_k} q^{-k}, \\ \int_{\mathbb{C}^*} |r_k^z(\gamma)|^2 d\sigma_z(\gamma) &= \frac{\theta(-1/z)}{\theta(-z/t_0 t_1, -z/t_0 t_2, -z/t_1 t_2)} \\ &\quad \cdot \frac{1}{(1/t_0 t_1, 1/t_0 t_2; q)_\infty^2} \frac{(-q^k z/t_0 t_1, -q^k z/t_0 t_2; q)_\infty}{(-q^k z/t_1 t_2, -q^{1+k} z; q)_\infty} q^{-k}. \end{aligned}$$

Proof. It follows from Theorem 9.1 that

$$\{\gamma \mapsto \phi_\gamma(-q^k) \mid k \in \mathbb{Z}_+\} \cup \{\gamma \mapsto \phi_\gamma(zq^k) \mid k \in \mathbb{Z}\}$$

is an orthogonal basis of $L_W^2(\mathbb{C}^*, d\nu)$, and that their quadratic norms are given by

$$\begin{aligned} \int_{\mathbb{C}^*} |\phi_\gamma(-q^k)|^2 d\nu(\gamma) &= \frac{p(-q^k)}{(1-q)q^k}, \quad k \in \mathbb{Z}_+, \\ \int_{\mathbb{C}^*} |\phi_\gamma(zq^k)|^2 d\nu(\gamma) &= \frac{p(zq^k)}{(1-q)zq^k}, \quad k \in \mathbb{Z}. \end{aligned}$$

The theorem follows now immediately by setting $t_0 = 1/a$, $t_1 = 1/b$ and $t_2 = 1/c$ and using the explicit expressions (2.6) and (8.7) of the function $p(\cdot)$ and the constant M . \square

Remark 9.3. We remarked in section 1 that the big q -Jacobi function transform is associated with harmonic analysis on the $SU(1,1)$ quantum group. Analogous considerations for the quantum group of plane motions lead to the so-called big q -Hankel transform, see [13] and [5]. The corresponding function theoretic aspects of the big q -Hankel transform are discussed in detail in [5].

The dual orthogonality relations for the big q -Hankel transform have a similar interpretation as the dual orthogonality relations for the big q -Jacobi function transform, namely, they give orthogonality relations for Moak's q -Laguerre polynomials with respect to a one-parameter family of non-extremal orthogonality measures, as well as explicit sets of functions which complement the q -Laguerre polynomials to orthogonal bases of the associated Hilbert spaces, see [5, Theorem 4.1].

10. BIG q -JACOBI POLYNOMIALS: FUNCTIONAL ANALYTIC APPROACH

We show in this section how the orthogonality relations and the quadratic norm evaluations for the big q -Jacobi polynomials can be derived from a functional analytic approach. The arguments are closely related to the ones used for the big q -Jacobi function transform, so we merely sketch the steps and indicate the main differences. To avoid confusion with previous notations, we label definitions in this section with a subscript (or superscript) \wp (indicating that it is connected with the polynomial case).

The conditions on the parameters a, b, c are now taken to be

$$ab, qa/b, ac, qa/c < 1, \quad bc < 0. \quad (10.1)$$

We consider the second order q -difference equation L acting on the space $\mathcal{F}(I_\varphi)$ of complex-valued functions $f : I_\varphi \rightarrow \mathbb{C}$, where $I_\varphi = [-1, -q/bc]_q = \{-q^k\}_{k \in \mathbb{Z}_+} \cup \{-q^{k+1}/bc\}_{k \in \mathbb{Z}_+}$. At the end-points $x = -1$ and $x = -q/bc$ this should be read as $(Lf)(x) = A(x)(f(qx) - f(x))$. We can write this in self-adjoint form, similarly as was done in Lemma 2.1 for the non-compact case, with the same functions p and r , see (2.6). Observe that $p(x) > 0$ for all $x \in I_\varphi$ by the conditions (10.1) on the parameters.

We define $\mathcal{H}_\varphi = \{f : I_\varphi \rightarrow \mathbb{C} \mid \|f\|_\varphi^2 = \langle f, f \rangle_\varphi < \infty\}$, where

$$\langle f, g \rangle_\varphi = \int_{-1}^{-q/bc} f(x) \overline{g(x)} \frac{d_q x}{p(x)}.$$

It is well known that the big q -Jacobi polynomials, which are explicitly given by (5.7), form an orthogonal basis of \mathcal{H}_φ , see [1]. This fact and the evaluation of the corresponding quadratic norms have been derived in [1] using the q -binomial formula [7, (1.3.2)] and the q -Pfaff-Saalschütz formula [7, (1.7.2)]. In this section we derive these results by functional analytic methods.

Similarly as in the non-compact setting, we truncate the inner product by

$$\langle f, g \rangle_{\varphi, k, l} = \left(\int_{-1}^{-q^{k+1}} + \int_{-q^{l+2}/bc}^{-q/bc} \right) f(x) \overline{g(x)} \frac{d_q x}{p(x)}$$

for $k, l \in \mathbb{Z}_+$. The analogue of Lemma 2.2 is then given by

$$\langle Lf, g \rangle_{\varphi, k, l} - \langle f, Lg \rangle_{\varphi, k, l} = W(f, \overline{g})(-q^k) - W(f, \overline{g})(-q^{l+1}/bc),$$

with the Wronskian as defined in (2.11).

Let $\alpha \in \mathbb{T}$, then we write $\mathcal{D}_{\varphi, \alpha}$ for the functions $f \in \mathcal{H}_\varphi$ such that $Lf \in \mathcal{H}_\varphi$ and $f(0^+) = \alpha f(0^-)$, $f'(0^+) = \alpha f'(0^-)$, where now $f(0^-) = \lim_{k \rightarrow \infty} f(-q^k)$ and $f(0^+) = \lim_{k \rightarrow \infty} f(-q^k/bc)$, etc. (cf. section 2). By similar arguments as in section 2, we have

Proposition 10.1. *The operator $(L, \mathcal{D}_{\varphi, \alpha})$ on \mathcal{H}_φ is self-adjoint for all $\alpha \in \mathbb{T}$.*

We denote $\mathcal{D}_\varphi = \mathcal{D}_{\varphi, 1}$. Now observe that $\phi_\gamma, \psi_\gamma \in \mathcal{D}_\varphi$ for $\gamma \in \mathbb{C}^*$ by Lemma 3.5 (see (3.6) and (3.7) for the definition of ϕ_γ and ψ_γ , respectively). Furthermore, from the arguments as given in section 3 it follows that $(L\phi_\gamma)(x) = \mu(\gamma)\phi_\gamma(x)$ for $x \in [-1, -q/bc]_q = I_\varphi \setminus \{-q/bc\}$ and $(L\psi_\gamma)(x) = \mu(\gamma)\psi_\gamma(x)$ for $x \in (-1, -q/bc]_q = I_\varphi \setminus \{-1\}$, where $\mu(\gamma)$ is given by (3.5). The Wronskian $W_\varphi(\gamma) = W(\psi_\gamma, \phi_\gamma) \in \mathcal{F}(I_\varphi)$ can again be seen to be constant on I_φ (cf. Lemma 3.1(iii)), and an explicit expression of the Wronskian $W_\varphi(\gamma)$ is given by

$$W_\varphi(\gamma) = (1 - q) \frac{(a\gamma, a/\gamma; q)_\infty \theta(bc)}{(ab, ac, qa/b, qa/c; q)_\infty}.$$

Indeed, observe that Proposition 5.1 is also valid for the present choice (10.1) of parameter values by analytic continuation.

In particular, the functions ϕ_γ and ψ_γ in \mathcal{D}_φ are linearly independent if and only if $\gamma \notin S_{pol}$, where $S_{pol} = \{\gamma_n^{\pm 1}\}_{n \in \mathbb{Z}_+}$, $\gamma_n = aq^n$. Hence $(L\phi_\gamma)(-q/bc) \neq \mu(\gamma)\phi_\gamma(-q/bc)$ and $(L\psi_\gamma)(-1) \neq \mu(\gamma)\psi_\gamma(-1)$ if $\gamma \notin S_{pol}$, cf. Corollary 5.2(iii).

Let $\mu = \mu(\gamma) \in \mathbb{C} \setminus \mathbb{R}$. We define the Green kernel $K_\gamma^\wp(x, y)$ for $x, y \in I_\wp$ by

$$K_\gamma^\wp(x, y) = \begin{cases} W_\wp(\gamma)^{-1} \psi_\gamma(x) \phi_\gamma(y), & y \leq x, \\ W_\wp(\gamma)^{-1} \phi_\gamma(x) \psi_\gamma(y), & y \geq x. \end{cases}$$

We have a well defined linear map $\mathcal{H}_\wp \rightarrow \mathcal{F}(I_\wp)$ which maps $f \in \mathcal{H}_\wp$ to

$$G_f^\wp(x, \gamma) = \langle f, \overline{K_\gamma^\wp(x, \cdot)} \rangle_\wp, \quad x \in I_\wp. \quad (10.2)$$

By similar arguments as in the proof of Proposition 6.1, we derive that for $f \in \mathcal{H}_\wp$ and for $\gamma \in \mathbb{C}^*$ such that $\mu(\gamma) \in \mathbb{C} \setminus \mathbb{R}$,

$$G_f^\wp(\cdot, \gamma) = (L - \mu(\gamma) \cdot \text{Id})^{-1} f. \quad (10.3)$$

For the proof of (10.3), we have to consider case 2 of the proof of Proposition 6.1 twice, namely for the end-point -1 as well as for the end-point $-q/bc$. The arguments go through for both end-points, since ϕ_γ (respectively ψ_γ) satisfies the eigenvalue equation $(Lf)(x) = \mu(\gamma)f(x)$ in the end-point $x = -1$ (respectively $x = -q/bc$).

We are now in a position to compute the resolution of the identity E_\wp for the self-adjoint operator (L, \mathcal{D}_\wp) on \mathcal{H}_\wp using (6.6). For the moment it is convenient to assume that the parameters also satisfy the condition $a^2 \notin \{q^k\}_{k \in \mathbb{Z}}$. This condition can be removed later on by continuity.

We keep the notations of the proof of Proposition 7.2. Choose $f, g \in \mathcal{H}_\wp$ with finite support. Then we have for $\mu \in \mathbb{R}$ and $\epsilon > 0$, that

$$\begin{aligned} \langle (L - (\mu \pm i\epsilon))^{-1} f, g \rangle_\wp &= \iint_{\substack{(x, y) \in I_\wp \times I_\wp \\ x \leq y}} \frac{\phi_{\gamma[\mu \pm i\epsilon]}(x) \psi_{\gamma[\mu \pm i\epsilon]}(y)}{W_\wp(\gamma[\mu \pm i\epsilon])} \left(1 - \frac{1}{2} \delta_{x, y}\right) \\ &\quad \cdot (f(x) \overline{g(y)} + f(y) \overline{g(x)}) \frac{d_q x}{p(x)} \frac{d_q y}{p(y)}. \end{aligned} \quad (10.4)$$

Let $\xi > 0$ such that $\mu(S_{pol}) \cap [-(1+a)^2 - \xi, -(1-a)^2 + \xi] = \emptyset$. Using (7.5), (8.11) and the invariance of ϕ_γ , ψ_γ and $W_\wp(\gamma)$ under $\gamma \leftrightarrow \gamma^{-1}$, we obtain from (10.4) that for all $-(1+a)^2 - \xi < \mu < -(1-a)^2 + \xi$,

$$\lim_{\epsilon \downarrow 0} \left(\langle (L - (\mu + i\epsilon))^{-1} f, g \rangle_\wp - \langle (L - (\mu - i\epsilon))^{-1} f, g \rangle_\wp \right) = 0.$$

It follows now from (6.6) that $E_\wp([-(1+a)^2, -(1-a)^2]) = 0$.

For $-\infty < \mu_1 < \mu_2 < -(1+a)^2$ or $-(1-a)^2 < \mu_1 < \mu_2 < \infty$ such that $(\mu_1, \mu_2) \cap \mu(S_{pol}) = \emptyset$, we have $E_\wp((\mu_1, \mu_2)) = 0$, cf. the proof of Proposition 8.3(i). Setting $\mu_n = \mu(\gamma_n)$ for $n \in \mathbb{Z}_+$, we have, due to the simple pole of $W_\wp(\gamma)^{-1}$ in $\gamma = \gamma_n$,

$$\begin{aligned} \langle E_\wp(\{\mu_n\}) f, g \rangle_\wp &= \iint_{\substack{(x, y) \in I_\wp \times I_\wp \\ x \leq y}} (f(x) \overline{g(y)} + f(y) \overline{g(x)}) \left(1 - \frac{1}{2} \delta_{x, y}\right) \\ &\quad \cdot \phi_{\gamma_n}(x) \psi_{\gamma_n}(y) \left(\frac{a}{\gamma_n} (\gamma_n^{-1} - \gamma_n) \text{Res}_{\gamma=\gamma_n} (W_\wp(\gamma)^{-1}) \right) \frac{d_q x}{p(x)} \frac{d_q y}{p(y)} \end{aligned}$$

for $f, g \in \mathcal{H}_\varphi$ with finite support, cf. the proof of Proposition 8.3(ii). Using (5.7) to rewrite ψ_{γ_n} as a multiple of ϕ_{γ_n} , and by symmetrizing the double q -Jackson integral, we obtain

$$\langle E_\varphi(\{\mu_n\})f, g \rangle_\varphi = \mathcal{N}_\varphi(n)^{-1} \langle f, \psi_{\gamma_n} \rangle_\varphi \langle \psi_{\gamma_n}, g \rangle_\varphi, \quad (10.5)$$

where

$$\begin{aligned} \mathcal{N}_\varphi(n) &= \left(\frac{a}{\gamma_n} (\gamma_n^{-1} - \gamma_n) \frac{(qa/b, qa/c; q)_n}{(ab, ac; q)_n} \left(\frac{bc}{q} \right)^n \operatorname{Res}_{\gamma=\gamma_n} (W_\varphi(\gamma)^{-1}) \right)^{-1} \\ &= \frac{(1-q)(q, bc, q/bc, q^{2n+1}a^2; q)_\infty (q, q^n a^2; q)_n}{(qa/b, qa/c, q^n ab, q^n ac; q)_\infty (qa/b, qa/c; q)_n} \left(-\frac{q^{(3-n)/2}}{bc} \right)^n. \end{aligned}$$

Observe that (10.5) holds for all $f, g \in \mathcal{H}_\varphi$ by continuity.

We can now immediately recover the orthogonality relations and quadratic norm evaluations for the big q -Jacobi polynomials by applying well known properties of resolution of identities to E_φ . The result is as follows.

Theorem 10.2 ([1]). *If the parameters (a, b, c) satisfy the conditions (10.1), then the polynomials $\{\psi_{\gamma_n}\}_{n \in \mathbb{Z}_+}$ form an orthogonal basis of \mathcal{H}_φ , and their quadratic norms are given by*

$$\|\psi_{\gamma_n}\|_\varphi^2 = \mathcal{N}_\varphi(n), \quad n \in \mathbb{Z}_+.$$

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